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[14] V. Gazi, B. Fidan, Y.S. Hanay, and M. Ilter Köksal, *Aggregation, Foraging and Formation Control of Swarm with Non-Holonomic Agents Using Potential Functions and Sliding Mode Techniques*, Turk. J. Elec. Engin. **15** (2) 2007.

[15] V. Gazi and R. Ordonez, Target Tracking Using Artificial Potentials and Sliding Mode Control, International Journal of Control, **80**(10) October 2007 1626-1635 .

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# 2-D SOLUTIONS IN RE-ENTRY AERODYNAMICS

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**Abstract**

We derive integrable solutions for the two-dimensional (2-D) re-entry dynamical equations of motion of a space vehicle, under the assumptions of standard atmospheric model. It is desirable to have analytical solutions for this important and practical problem which arise during the atmospheric re-entry phase. Therefore, our solution can be effectively applied to investigate and control the rocket flight characteristics. By setting the initial conditions for the speed, re-entering flight-path angle, altitude, atmosphere density, lift and drag coefficients, the nonlinear differential equations of motion are linearized by a proper choice of the re-entry range angles. By carrying out the closed-form integration, we express the solutions with the Exponential Integral, and Generalized Exponential Integral functions. Theoretical frameworks for proposed solutions as well as, several numerical examples, are presented.

## 1. Introduction

Since the beginning of the space flight, one of the most important aerodynamic problem encountered in astronautics is the return of satellites and space vehicles

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to the Earth's surface.

Inaccurate knowledge of the flight characteristics may lead to unacceptable errors in trajectory stability during the guidance and control process. This requires analytical solutions to the re-entry dynamical equations of motion of space vehicle, during the atmospheric re-entry stage.

The main goal of this paper is to derive approximate analytical solutions for the equations of trajectory and flight-path angle of a space vehicle during the re-entry phase in Earth's atmosphere.

Information about re-entry trajectories can be instantly obtained if the space vehicle's dynamical equations are integrated in closed form. In practice, traditional algorithms for numerical integration, see Regan [1] or Press et al. [2] can be alternatively employed to approximate the solutions and compare it with the analytical results.

Here we derive explicit solutions for the two-dimensional (2-D) equations of motion, known in the literature as planar motion [3, 4], when we know the initial conditions for the atmospheric re-entry parameters such as velocity, angle, altitude, atmospheric density, lift and drag coefficients.

Under the assumptions of the Standard Atmosphere Model, i.e. the density varying exponentially with the altitude, the Newton's second law completely describes the re-entry dynamical equations of motion.

The re-entry problem becomes very complex when the analytical integration of a system of two nonlinear differential equation is no longer possible.

No many analytical closed-form solutions are well known, most of them developed during the period of 1960-1970's, e.g. see Ehrike [3], Hankey [5], Regan [6], Loh [7] and Dunning's [8, 9] NASA reports, and more recently by one of the author, see Mititelu [16].

The paper is organized as following. In Section we presents the basic dynamics of re-entry, as well as the differential equations of motion. In section , under some simple approximations, the nonlinear differential equations of motion are reduced to standard ordinary non-homogeneous linear differential equations, and integrated in closed forms.

## 2. The planar equations of motion during re-entry stage

It can be shown (see Mititelu [16] or Dunning's [8, 9]) that for a space vehicle at the initial re-entry point on its trajectory at altitude $z_i$ above the Earth surface with the initial velocity $V_i$, and initial flight path-angle $\theta_i$, the governing equations of motion in the Earth's axes coordinates are

$$m\dot{V}(t) = -\frac{\rho V^2(t)}{2} SC_D - mg \sin\theta(t) \tag{1}$$



respectively

$$mV(t)\dot{\theta}(t) = \frac{\rho V^2(t)}{2} SC_L - m\left(g - \frac{V^2(t)}{R_0}\right)\cos\theta(t) \tag{2}$$

where $C_D$ and $C_L$ denotes the drag and the lift coefficients, and $\rho(t)$ is the Earth atmosphere density at the time $t$ after re-entry.

The scalar components for velocity projections along the Earth-axes are:

$$\begin{aligned} V_z(t) &= \dot{z}(t) = V(t)\sin\theta(t) \\ V_x(t) &= \dot{x}(t) = V(t)\cos\theta(t) \end{aligned} \tag{3}$$

where $\theta(t)$ is the satellite re-entry angle during manoeuvre at time $t$.

According to the standard conventions, the path-angles are negative for re-entering and positive if the vehicle is moving up on the trajectory.

Therefore, in the case $\theta(t) < 0$ substitute $\theta(t)$ with $-\theta(t)$ the second relation on Eq.(3) remain unchanged, the first relation on Eq.(3) is changing to $V_z(t) = -V(t)\sin\theta(t)$, and equations Eq.(1), Eq.(2) becomes

$$m\dot{V}(t) = -\frac{\rho V^2(t)}{2} SC_D + mg \sin\theta(t) \tag{4}$$

$$mV(t)\dot{\theta}(t) = -\frac{\rho V^2(t)}{2} SC_L + m\left(g - \frac{V^2(t)}{R_0}\right)\cos\theta(t) \tag{5}$$

with the substitution $\theta(t) = \frac{\pi}{2} - \varphi(t)$, $\dot{\theta}(t) = -\dot{\varphi}(t)$, the equations of motions may be found in the same form in Dunning's paper [9]

$$m\dot{V}(t) = -\frac{\rho V^2(t)}{2} SC_D + mg \cos\varphi(t) \tag{6}$$

$$mV(t)\dot{\varphi}(t) = \frac{\rho V^2(t)}{2} SC_L - m\left(g - \frac{V^2(t)}{R_0}\right)\sin\varphi(t) \tag{7}$$

The equations Eq.(6) and Eq.(7) can be expressed via a new re-scaled variable $V(t)/\sqrt{gR_0}$, in the form

$$\sqrt{\frac{R_0}{g}}\frac{d}{dt}\left(\frac{V(t)}{\sqrt{gR_0}}\right) = -\frac{1}{2}\left(\frac{SC_D}{m}\right)\rho R_0\left(\frac{V^2(t)}{gR_0}\right) + \cos\varphi(t) \tag{8}$$

$$-\sqrt{\frac{R_0}{g}}\,\dot{\varphi}(t) = \left[\frac{1}{\left(\frac{V(t)}{\sqrt{gR_0}}\right)} - \left(\frac{V(t)}{\sqrt{gR_0}}\right)\right]\sin\varphi(t) - \frac{1}{2}\left(\frac{SC_L}{m}\right)\rho R_0\left(\frac{V(t)}{\sqrt{gR_0}}\right) \tag{9}$$

In the approximation of isothermal atmosphere [6] the variation of density $\rho$ with the altitude $z$ can be expressed with the Barometric Law as

$$\rho(z) = \rho_0 e^{-\beta z} \tag{10}$$

where $\rho_0$ is the density at $z = 0$ on Earth surface (sea level density), and $\beta$ is the barometric coefficient. Differentiating the Barometric Law and use the velocity expression along $z$ axis, $V_z(t) = \dot{z}(t) = -V(t)\sin\theta(t) = -V(t)\cos\varphi(t)$, then

$$d\rho(z(t)) = \beta\rho(z(t))V(t)\cos\varphi(t)\,dt \tag{11}$$

and Eq.(8), Eq.(9) expressed in the variable $\rho$ becomes

$$\frac{d}{d\rho}\left(\frac{V^2(\rho)}{gR_0}\right) + \left(\frac{SC_D}{m\beta}\right)\frac{(V^2(\rho)/gR_0)}{\cos\varphi(\rho)} = \frac{1}{\rho}\left(\frac{2}{\beta R_0}\right) \tag{12}$$

$$\frac{d(\sin\varphi(\rho))}{d\rho} + \left(\frac{1}{\beta R_0}\right)\frac{\sin\varphi(\rho)}{\rho}\left(\frac{gR_0}{V^2(\rho)} - 1\right) = \frac{1}{2}\left(\frac{SC_L}{m\beta}\right) \tag{13}$$

In the next section certain closed form solutions for the above equations may be evaluated under various initial conditions. We derive the expression for the space vehicle velocity $V(t)$ after re-entry, respectively the fight path-angle $\varphi(t)$ during the motion on the re-entry trajectory, at two different stages.

In Section we analyze the case of small re-entry angles, chosen such that in the range, let said $1^{deg} \leq |\varphi(t)| \leq 10^{deg}$ we can perform some approximations and integrate the system of differential equations Eq.(8), Eq.(9) to obtain the analytical solutions for $V(t)$ and $\varphi(t)$.

In Section we analyze the case of large re-entry angles $\varphi$, considered for practical reasons, in the interval $10^{deg} \leq |\varphi(t)| \leq 30^{deg}$, and integrate the differential equations Eq.(12), Eq.(13) in closed-form. As pointed out by Hankey [5] and Frank [1] for practical purpose analytical solutions are preferred for rapid estimation of the design parameters.

## 3. Analytical solutions for the re-entry equations

### 3.1 The case of small flying path-angles

With the change of variable $y(t) = V(t)/\sqrt{gR_0}$ and introducing the notations:

$$\tau = \sqrt{\frac{R_0}{g}},\ \mu_D = \left(\frac{SC_D}{2m}\right)\rho R_0,\ \mu_L = \left(\frac{SC_L}{2m}\right)\rho R_0 \tag{14}$$



where $\mu_D$ and $\mu_L$ are real positive dimensionless constants, the nonlinear differential equations Eq.(8), and Eq.(9) becomes

$$\tau\dot{y}(t) = -\mu_D\, y^2(t) + \cos\varphi(t) \tag{15}$$

$$-\tau\dot{\varphi}(t) = -\mu_L\, y(t) + \left(\frac{1 - y^2(t)}{y(t)}\right)\sin\varphi(t) \tag{16}$$

Expanding $\sin\varphi(t)$ and $\cos\varphi(t)$ in the power series for small values of the angle $\varphi(t)$ (see [11])

$$\begin{aligned} \sin\varphi(t) &= \sum_{n=0}^{\infty}(-1)^n\frac{\varphi^{2n+1}(t)}{(2n+1)!} \approx \varphi(t) \\ \cos\varphi(t) &= \sum_{n=0}^{\infty}(-1)^n\frac{\varphi^{2n}(t)}{(2n)!} \approx 1 \end{aligned} \tag{17}$$

and truncate these series to the first terms corresponding to the first order approximation the equations Eq.(15), Eq.(16) becomes:

$$\tau\dot{y}(t) = -\mu_D\, y^2(t) + 1 \tag{18}$$

$$-\tau\dot{\varphi}(t) = -\mu_L\, y(t) + \left(\frac{1 - y^2(t)}{y(t)}\right)\varphi(t) \tag{19}$$

Separate the variables in the equation Eq.(18) and integrate from the initial re-entry value $y_i = y(t_i) = V_i/\sqrt{gR_0}$ to the final value $y(t)$, and using the well known exact expression for the integral $\int(a^2 - x^2)^{-1}dx = \frac{1}{a}\,\text{arctanh}\left(\frac{x}{a}\right)$, for $a > 0$ (see [12]), the final solution is

$$\begin{aligned} y(t) &= \frac{1}{\sqrt{\mu_D}}\tanh\left[\frac{\sqrt{\mu_D}}{\tau}(t - t_i) + \text{arctanh}\,(y_i\sqrt{\mu_D})\right] \\ &= \frac{y_i + \frac{1}{\sqrt{\mu_D}}\tanh\left[\frac{\sqrt{\mu_D}}{\tau}(t - t_i)\right]}{1 + y_i\sqrt{\mu_D}\tanh\left[\frac{\sqrt{\mu_D}}{\tau}(t - t_i)\right]} \end{aligned} \tag{20}$$

The Eq.(20) may be written as:

$$\begin{aligned} y(t) &= \frac{1}{\sqrt{\mu_D}}\tanh\left[t\left(\frac{\sqrt{\mu_D}}{\tau}\right) + \kappa\right] \\ \kappa &= \text{arctanh}\,(y_i\sqrt{\mu_D}) - t_i\frac{\sqrt{\mu_D}}{\tau} \end{aligned} \tag{21}$$

Table 1: **Re-entry velocity at small flying path-angles when the initial re-entry parameters are specified**

| $V$ (Km/sec) | $SC_D/m$ | $t_i$ (sec) | $z_i$ (Km) | $V_i$ (Km/sec) | $t$ (sec) |
|---|---|---|---|---|---|
| 14.87 | 0.1 | 5 | 95 | 15 | 30 |
| 10.29 | 0.15 | 10 | 125 | 10 | 40 |
| 6.79 | 0.2 | 2 | 85 | 8 | 60 |
| 21.02 | 0.25 | 8 | 10 | 25 | 130 |
| 32.41 | 0.3 | 20 | 110 | 35 | 180 |
| 18.91 | 0.4 | 12 | 105 | 20 | 90 |
| 8.95 | 0.5 | 15 | 90 | 30 | 140 |

Now, taking into account the notations Eq.(14), Eq.(21) becomes

$$V(t) = \sqrt{\left(\frac{2g}{\rho(z)}\right)\left(\frac{m}{SC_D}\right)} \tanh\left[t\sqrt{\left(\frac{g}{2}\right)\left(\frac{SC_D}{m}\right)\rho(z)} + \kappa\right]$$

$$= \sqrt{\left(\frac{2g}{\rho_0}\right)\left(\frac{m}{SC_D}\right)e^{\beta z}} \tanh\left[t\sqrt{\left(\frac{g\rho_0}{2}\right)\left(\frac{SC_D}{m}\right)e^{-\beta z}} + \kappa\right] \quad (22)$$

Substitute the solution Eq.(21) in Eq.(19) the time variation of re-entering path angle may be written as

$$\frac{d\varphi(t)}{dt} + \frac{1}{\tau}\left[-\frac{1}{\sqrt{\mu_D}}\tanh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right) + \sqrt{\mu_D}\,\mathrm{cotanh}\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)\right]\varphi(t)$$
$$= \frac{\mu_L}{\tau\sqrt{\mu_D}}\tanh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right) \quad (23)$$

The Eq.(23)is a non-homogeneous linear differential equation of the form:

$$\frac{d\varphi(t)}{dt} + P(t)\varphi(t) = Q(t) \quad (24)$$

where $P(t)$, $Q(t)$ are continuous functions on the interval $[t_i, t]$, with the general solution

$$\varphi(t) = \exp\left(-\int_{t_i}^{t} P(t)dt\right)\int_{t_i}^{t} Q(t)\exp\left(\int_{t_i}^{t} P(t)dt\right)dt \quad (25)$$

with the initial condition $\varphi(t_i) = 0$. The functions $P(t)$ and $Q(t)$ corresponding



to Eq.(23) are:

$$P(t) = -\frac{1}{\tau\sqrt{\mu_D}}\tanh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right) + \frac{\sqrt{\mu_D}}{\tau}\mathrm{cotanh}\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)$$

$$Q(t) = \frac{\mu_L}{\tau\sqrt{\mu_D}}\tanh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right) \quad (26)$$

then

$$\int_{t_i}^{t} P(t)dt = \log\left[\frac{\sinh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)}{\sinh\left(t_i\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)}\right] + \log\left\{\left[\frac{\cosh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)}{\cosh\left(t_i\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)}\right]^{-\frac{1}{\mu_D}}\right\} \quad (27)$$

and

$$\exp\left(-\int_{t_i}^{t} P(t)dt\right)dt = K\frac{\left[\cosh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)\right]^{\frac{1}{\mu_D}}}{\sinh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)} \quad (28)$$

where $K$ denotes the constant term

$$K = \frac{\sinh\left(t_i\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)}{\left[\cosh\left(t_i\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)\right]^{1/\mu_D}} \quad (29)$$

Substitute Eq.(26), Eq.(27), Eq.(28) in Eq.(25) and simplify the relation with the constant $K$ after some algebra we obtain

$$\varphi(t) = \frac{\mu_L}{\tau\sqrt{\mu_D}}\frac{\left[\cosh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)\right]^{\frac{1}{\mu_D}}}{\sinh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)}\int_{t_i}^{t}\tanh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)\frac{\sinh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)}{\left[\cosh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)\right]^{1/\mu_D}}dt$$

$$= \left(\frac{\mu_L}{\tau\sqrt{\mu_D}}\right)\frac{\left[\cosh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)\right]^{\frac{1}{\mu_D}}}{\sinh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)}$$
$$\cdot\left\{\int_{t_i}^{t}\left[\cosh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)\right]^{-\frac{1}{\mu_D}+1}dt - \int_{t_i}^{t}\left[\cosh\left(t\frac{\sqrt{\mu_D}}{\tau} + \kappa\right)\right]^{-\frac{1}{\mu_D}-1}dt\right\} \quad (30)$$

Using the integrals of the following type (see Annex A, Theorem 1):

$$
\begin{aligned}
I_{1-p}(t) &= \int_{t_i}^{t} \cosh^{1-p}(At+\kappa)dt \\
&= \frac{1}{2^{2-p}A}\mathrm{Re}\left\{(-1)^{\frac{3-p}{2}}\ \mathbb{B}\left[1+e^{2(At_i+\kappa)}, 1+e^{2(At+\kappa)}, 2-p, \frac{p-1}{2}\right]\right\} \\
I_{-1-p}(t) &= \int_{t_i}^{t} \cosh^{-1-p}(At+\kappa)dt \\
&= \frac{1}{2^{-p}A}\mathrm{Re}\left\{(-1)^{\frac{1-p}{2}}\ \mathbb{B}\left[1+e^{2(At_i+\kappa)}, 1+e^{2(At+\kappa)}, -p, \frac{p+1}{2}\right]\right\}
\end{aligned}
\tag{31}
$$

where $\mathrm{Re}\{\cdot\}$ denotes the real part, $p \in \mathbb{R}_+$ is a positive real number and

$$
\mathbb{B}[z_0, z_1; a, b] = \int_{z_0}^{z_1} t^{a-1}(1-t)^{b-1}dt \tag{32}
$$

is the Generalized Incomplete Beta Function(see [12] or [11]), where here

$$
\begin{aligned}
A &= \frac{\sqrt{\mu_D}}{\tau} = \sqrt{\frac{g}{2}\left(\frac{SC_D\rho}{m}\right)} \\
\kappa &= \operatorname{arctanh}\left[V_i\sqrt{\frac{1}{2g}\left(\frac{SC_D\rho}{m}\right)}\right] - t_i\sqrt{\frac{g}{2}\left(\frac{SC_D\rho}{m}\right)} \\
p &= \frac{1}{\mu_D} = \frac{2}{R_0}\left(\frac{m}{SC_D}\right)\frac{1}{\rho}
\end{aligned}
\tag{33}
$$

By combining Eq.(30) and Eq.(31) may produce the following solution for the re-entry angle

$$
\varphi(t) = \left(\frac{\mu_L}{\mu_D}\right)\frac{\left\{2\cosh\left[t\left(\frac{\sqrt{\mu_D}}{\tau}\right)+\kappa\right]\right\}^{\frac{1}{\mu_D}}}{\sinh\left[t\left(\frac{\sqrt{\mu_D}}{\tau}\right)+\kappa\right]}
$$

$$
\begin{aligned}
&\mathrm{Re}\left\{(-1)^{\frac{3\mu_D-1}{2\mu_D}}\left\{\frac{1}{4}\,\mathbb{B}\left[1+e^{2\left(t_i\left(\frac{\sqrt{\mu_D}}{\tau}\right)+\kappa\right)}, 1+e^{2\left(t\left(\frac{\sqrt{\mu_D}}{\tau}\right)+\kappa\right)}, 2-\frac{1}{\mu_D}, \frac{1-\mu_D}{2\mu_D}\right]\right.\right. \\
&\left.\left.+\,\mathbb{B}\left[1+e^{2\left(t_i\left(\frac{\sqrt{\mu_D}}{\tau}\right)+\kappa\right)}, 1+e^{2\left(t\left(\frac{\sqrt{\mu_D}}{\tau}\right)+\kappa\right)}\ -\frac{1}{\mu_D}, \frac{1+\mu_D}{2\mu_D}\right]\right\}\right\}
\end{aligned}
\tag{34}
$$



Table 2: **Re-entry flying path-angles for specified re-entry initial conditions**

| $\varphi$ (deg) | $SC_D/m$ | $t_i$ (sec) | $z_i$ (Km) | $V_i$ (Km/sec) | $t$ (sec) |
|---|---|---|---|---|---|
| 8.44° | 0.1 | 10 | 100 | 12 | 25 |
| 7.35° | 0.15 | 10 | 90 | 14 | 65 |
| 8.81° | 0.2 | 50 | 95 | 25 | 55 |
| 3.51° | 0.25 | 120 | 110 | 30 | 60 |
| 5.84° | 0.3 | 100 | 120 | 35 | 20 |
| 2.91° | 0.4 | 110 | 115 | 40 | 60 |
| 1.53° | 0.5 | 70 | 120 | 45 | 10 |

## 3.2 The case of large flying path-angles

These re-entry path angels are valid with the approximation that $1/\cos\varphi(\rho) \approx 1$, see Alexe and Staicu [13]. Performing a change of variable $y(\rho) = V^2(\rho)/gR_0$, the Eq.(12) becomes

$$
\frac{dy(\rho)}{d\rho} + \frac{SC_D}{m\beta}y(\rho) = \frac{1}{\rho}\left(\frac{2}{\beta R_0}\right) \tag{35}
$$

The solution of Eq.(35) with the initial condition $y_i = y(\rho_i) = V^2(\rho_i)/gR_0 = V_i^2/gR_0$ is given by

$$
\begin{aligned}
y(\rho) &= \exp\left(-\int_{\rho_i}^{\rho}\frac{SC_D}{m\beta}d\rho\right)\left[\int_{\rho_i}^{\rho}\frac{1}{\rho}\left(\frac{2}{\beta R_0}\right)e^{\frac{SC_D}{m\beta}(\rho-\rho_i)}d\rho + y_i\right] \\
&= \frac{2}{\beta R_0}e^{-\left(\frac{SC_D}{m\beta}\right)\rho}\int_{\rho_i}^{\rho}\frac{e^{\left(\frac{SC_D}{m\beta}\right)\rho}}{\rho}d\rho + y_i e^{-\frac{SC_D}{m\beta}(\rho-\rho_i)}
\end{aligned}
\tag{36}
$$

and the integral in the last term of Eq.(36) is one of the following type (see Annex A, Theorem 2)

$$
\int_{\rho_i}^{\rho}\frac{e^{\alpha\rho}}{\rho}d\rho = \mathbf{Ei}(\alpha\rho) - \mathbf{Ei}(\alpha\rho_i), \quad \alpha = \frac{SC_D}{m\beta} \tag{37}
$$

where $\mathbf{Ei}(x)$ is the Exponential Integral Function defined in the case of $x > 0$ as the Cauchy Principal Value

$$
\mathbf{Ei}(x) = -\lim_{x\to 0^+}\left[\int_{-x}^{-\varepsilon}\frac{e^{-t}}{t}dt + \int_{\varepsilon}^{+\infty}\frac{e^{-t}}{t}dt\right] \tag{38}
$$

Thus, the dependence of velocity $V(t)$ on density $\rho$, when the initial re-entry density $\rho_i$ and velocity $V_i$ are known, is

$$\frac{V^2(\rho)}{gR_0} = \left(\frac{2}{\beta R_0}\right) e^{-\left(\frac{SC_D}{m\beta}\right)\rho} \left[\mathbf{Ei}\left(\frac{SC_D}{m\beta}\rho\right) - \mathbf{Ei}\left(\frac{SC_D}{m\beta}\rho_i\right)\right] + \frac{V_i^2}{gR_0} e^{-\frac{SC_D}{m\beta}(\rho-\rho_i)} \tag{39}$$

or

$$V^2(z, SC_D/m) = \left(\frac{2g}{\beta}\right) e^{-\left(\frac{SC_D}{m\beta}\right)\rho_0 \exp(-\beta z)} \left[\mathbf{Ei}\left(\frac{SC_D}{m\beta}\rho_0 \exp(-\beta z)\right) - \mathbf{Ei}\left(\frac{SC_D}{m\beta}\rho_i\right)\right] + \frac{V_i^2}{gR_0} e^{-\frac{SC_D}{m\beta}[\rho_0 \exp(-\beta z) - \rho_i]} \tag{40}$$

Table 3: **Re-entry velocity at large flying path-angles when the initial re-entry parameters are specified**

| $V$ (Km/sec) | $V_i$ (Km/sec) | $SC_D/m$ | $z_i$ (sec) | $z$ (Km) |
|---|---|---|---|---|
| 1.2 | 105 | 0.1 | 150 | 75 |
| 0.4 | 145 | 0.2 | 110 | 45 |
| 0.95 | 100 | 0.3 | 120 | 65 |
| 1.26 | 130 | 0.4 | 160 | 80 |
| 1.17 | 150 | 0.5 | 180 | 60 |
| 1.31 | 190 | 0.6 | 175 | 85 |

With a change of variables

$$u(\rho) = \sin\varphi(\rho), \quad a_L = \frac{1}{2}\left(\frac{SC_L}{m\beta}\right)$$
$$b(\rho) = \frac{1}{\beta R_0}\left(\frac{gR_0}{V^2(\rho)} - 1\right) \tag{41}$$

Eq.(13) becomes

$$\frac{du(\rho)}{d\rho} + \frac{b(\rho)}{\rho}u(\rho) = a_L \tag{42}$$

The next step is to solve the Eq.(42) under the initial condition $u(\rho_i) = u_i = \sin\varphi_i$. Substitute $gR_0/V^2(\rho)$ from Eq.(39) in the last equality of Eq.(41) one obtain

$$b(\rho) = \frac{1}{2}\frac{e^{2\rho a_D}}{\mathbf{Ei}(2\rho a_D) + C} - \frac{1}{\beta R_0} \tag{43}$$

where

$$a_D = \frac{1}{2}\left(\frac{SC_D}{m\beta}\right) \quad \text{and } C = \frac{V_i^2\beta}{2g}e^{2\rho_i a_D} - \mathbf{Ei}(2\rho_i a_D) \tag{44}$$



The general solution of Eq.(42) is

$$u(\rho) = \exp\left(-\int_{\rho_i}^{\rho}\frac{b(\rho)}{\rho}d\rho\right)\left[\int_{\rho_i}^{\rho} a_L \exp\left(\int_{\rho_i}^{\rho}\frac{b(\rho)}{\rho}d\rho\right)d\rho + u(\rho_i)\right] \tag{45}$$

and,

$$\int_{\rho_i}^{\rho}\frac{b(\rho)}{\rho}d\rho = \frac{1}{2}\int_{\rho_i}^{\rho}\frac{e^{2\rho a_D}/\rho}{\mathbf{Ei}(2\rho a_D) + C}d\rho - \frac{1}{\beta R_0}\ln\left(\frac{\rho}{\rho_i}\right) \tag{46}$$

Using the integral of the following type (see Annex A, Theorem 3)

$$I_{m,n} = \int_a^b \frac{e^{mx}}{x}\frac{dx}{n + \mathbf{Ei}(mx)} = \ln\left[\frac{n + \mathbf{Ei}(mb)}{n + \mathbf{Ei}(ma)}\right] \tag{47}$$

$$\int_{\rho_i}^{\rho}\frac{b(\rho)}{\rho}d\rho = \ln\left[\left(\frac{\rho}{\rho_i}\right)^{-1/\beta R_0}\sqrt{\frac{C + \mathbf{Ei}(2\rho a_D)}{C + \mathbf{Ei}(2\rho_i a_D)}}\right] \tag{48}$$

and

$$\int_{\rho_i}^{\rho} a_L \exp\left(\int_{\rho_i}^{\rho}\frac{b(\rho)}{\rho}d\rho\right)d\rho = \frac{a_L\sqrt{C}\rho_i^{1/\beta R_0}}{\sqrt{C + \mathbf{Ei}(2\rho_i a_D)}}\int_{\rho_i}^{\rho}\rho^{-1/\beta R_0}\left(\sqrt{1 + \frac{\mathbf{Ei}(2\rho a_D)}{C}}\right)d\rho \tag{49}$$

In Eq.(46) using the series expansion (see [12])

$$\sqrt{1 + \frac{\mathbf{Ei}(2\rho a_D)}{C}} = 1 + \frac{1}{2}\left(\frac{\mathbf{Ei}(2\rho a_D)}{C}\right) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n-3)!!}{(2n)!!}\left(\frac{\mathbf{Ei}(2\rho a_D)}{C}\right)^n$$
$$\approx 1 + \frac{1}{2}\left(\frac{\mathbf{Ei}(2\rho a_D)}{C}\right) \tag{50}$$

where the series in the Eq.(50) is truncate at the first two terms. Notice that the expansion Eq.(50) is valid only if $|\ \mathbf{Ei}(2\rho a_D)/C\ | < 1$, where $(2n-3)!! = 1\cdot 3\cdot 5\ldots(2n-3)$ and $(2n)!! = 2\cdot 4\cdot 6\ldots(2n)$, is the double factorial. For the general discussion regarding the validity of expansion Eq.(50), see Mititelu [16].

Substitute the approximation Eq.(50) in Eq.(49), then

$$\int_{\rho_i}^{\rho} \rho^{-1/\beta R_0}\left(\sqrt{1+\frac{\mathbf{Ei}(2\rho a_D)}{\mathcal{C}}}\right)d\rho \simeq \int_{\rho_i}^{\rho} \rho^{-1/\beta R_0}\left(1+\frac{1}{2}\frac{\mathbf{Ei}(2\rho a_D)}{\mathcal{C}}\right)d\rho$$

$$=\frac{1}{[1-(1/\beta R_0)]}\left[\rho^{1-(1/\beta R_0)}-\rho_i^{1-(1/\beta R_0)}\right]+\frac{1}{2\mathcal{C}}\int_{\rho_i}^{\rho}\rho^{-1/\beta R_0}\,\mathbf{Ei}(2\rho a_D)d\rho \tag{51}$$

The last integral in Eq.(51) is one of the following type

$$\int_{\rho_i}^{\rho}\rho^{-1/\beta R_0}\,\mathbf{Ei}(2\rho a_D)d\rho=\frac{1}{[1-(1/\beta R_0)]}\left\{\rho_i^{1-(1/\beta R_0)}[\mathbf{E}_{1/\beta R_0}(-2\rho_i a_D)+\mathbf{Ei}(2\rho_i a_D)]\right.$$
$$\left.-\,\rho^{1-(1/\beta R_0)}\left[\mathbf{E}_{1/\beta R_0}(-2\rho a_D)+\mathbf{Ei}(2\rho a_D)\right]\right\} \tag{52}$$

where $\mathbf{E}_p(z)$ denotes the Generalized Exponential Integral Function. By combining Eq.(52) and Eq.(51) one obtain

$$\int_{\rho_i}^{\rho}\rho^{-1/\beta R_0}\left(\sqrt{1+\frac{\mathbf{Ei}(2\rho a_D)}{\mathcal{C}}}\right)d\rho=\frac{1}{[1-(1/\beta R_0)]}$$
$$\cdot\left\{\rho^{1-(1/\beta R_0)}\left[1+\frac{\mathbf{E}_{1/\beta R_0}(-2\rho a_D)+\mathbf{Ei}(2\rho a_D)}{2\mathcal{C}}\right]\right. \tag{53}$$
$$\left.-\rho_i^{1-(1/\beta R_0)}\left[1+\frac{\mathbf{E}_{1/\beta R_0}(-2\rho_i a_D)+\mathbf{Ei}(2\rho_i a_D)}{2\mathcal{C}}\right]\right\}$$

Based on Eq.(53), Eq.(49) becomes

$$\int_{\rho_i}^{\rho}a_L\exp\left(\int_{\rho_i}^{\rho}\frac{b(\rho)}{\rho}\right)d\rho=\frac{a_L\rho_i^{1/\beta R_0}}{[1-(1/\beta R_0)]\sqrt{1+[\mathbf{Ei}(2\rho_i a_D)/\mathcal{C}]}}$$
$$\cdot\left\{\rho^{1-(1/\beta R_0)}\left[1+\frac{\mathbf{E}_{1/\beta R_0}(-2\rho a_D)+\mathbf{Ei}(2\rho a_D)}{2\mathcal{C}}\right]\right. \tag{54}$$
$$\left.-\rho_i^{1-(1/\beta R_0)}\left[1+\frac{\mathbf{E}_{1/\beta R_0}(-2\rho_i a_D)+\mathbf{Ei}(2\rho_i a_D)}{2\mathcal{C}}\right]\right\}$$

Substitute Eq.(54) and Eq.(48) in Eq.(45) the final solution can be expressed as



$$u(\rho)=\left(\frac{\rho}{\rho_i}\right)^{1/\beta R_0}\sqrt{\frac{\mathcal{C}+\mathbf{Ei}(2\rho_i a_D)}{\mathcal{C}+\mathbf{Ei}(2\rho a_D)}}\left\{u(\rho_i)+\frac{a_L\rho_i^{1/\beta R_0}}{[1-(1/\beta R_0)]\sqrt{1+[\mathbf{Ei}(2\rho_i a_D)/\mathcal{C}]}}\right.$$
$$\cdot\left\{\rho^{1-(1/\beta R_0)}\left[1+\frac{\mathbf{E}_{1/\beta R_0}(-2\rho a_D)+\mathbf{Ei}(2\rho a_D)}{2\mathcal{C}}\right]\right.$$
$$\left.\left.-\rho_i^{1-(1/\beta R_0)}\left[1+\frac{\mathbf{E}_{1/\beta R_0}(-2\rho_i a_D)+\mathbf{Ei}(2\rho_i a_D)}{2\mathcal{C}}\right]\right\}\right\} \tag{55}$$

Returning now to the change of variable given by Eq.(41), $\varphi(\rho)=\arcsin(u(\rho))$, and introducing the notations

$$\mathcal{B}=\frac{a_L\rho_i\sqrt{\mathcal{C}}}{[1-(1/\beta R_0)]}=\frac{\rho_i}{[1-(1/\beta R_0)]}\left(\frac{1}{2}\frac{SC_L}{m\beta}\right)\sqrt{\frac{V_i^2\beta}{2g}\exp\left(\frac{SC_D}{m\beta}\rho_i\right)-\mathbf{Ei}\left(\frac{SC_D}{m\beta}\rho_i\right)} \tag{56}$$

$$\mathcal{D}=1+\frac{\mathbf{E}_{1/\beta R_0}\left(-\frac{SC_D}{m\beta}\rho_i\right)+\mathbf{Ei}\left(\frac{SC_D}{m\beta}\rho_i\right)}{(V_i^2\beta/g)\exp\left(\frac{SC_D}{m\beta}\rho_i\right)-2\,\mathbf{Ei}\left(\frac{SC_D}{m\beta}\rho_i\right)} \tag{57}$$

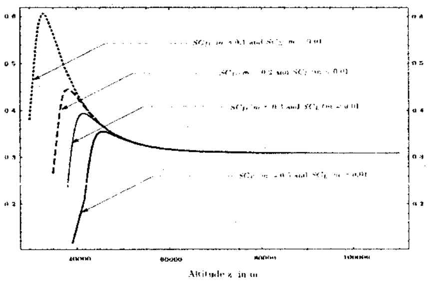
then for fixed values of initial re-entry altitude $z_i$, and of ratios $SC_D/m$, $SC_L/m$ the quantities $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, remain constants during the motion. Substitute Eq.(56) and Eq.(57) in Eq.(55) the final solution of Eq.(13), which gives the characteristics of the re-entry flight path-angle, is

$$\sin\varphi(\rho(z))=\left(\frac{\rho(z)}{\rho_i}\right)^{1/\beta R_0}\frac{1}{\sqrt{\mathcal{C}+\mathbf{Ei}\left(\frac{SC_D}{m\beta}\rho(z)\right)}}\left\{(V_i\sin\varphi_i)\sqrt{\frac{\beta}{2g}\exp\left(\frac{SC_D}{m\beta}\rho_i\right)}\right.$$
$$\left.+\mathcal{B}\left[\left(\frac{\rho(z)}{\rho_i}\right)^{1-(1/\beta R_0)}\cdot\left[1+\frac{\mathbf{E}_{1/\beta R_0}\left(-\frac{SC_D}{m\beta}\rho(z)\right)+\mathbf{Ei}\left(\frac{SC_D}{m\beta}\rho(z)\right)}{2\mathcal{C}}\right]-\mathcal{D}\right]\right\} \tag{58}$$

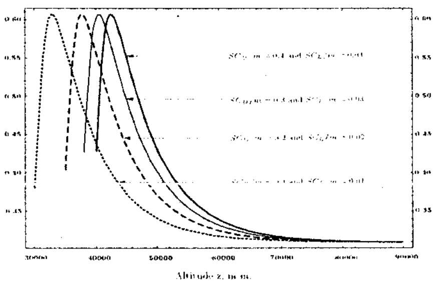
where $\mathcal{A}=V_i\sin\varphi_i\sqrt{\beta/2g}$, is a constant factor which depends on initial re-entry velocity and path-angle. The general solution Eq.(58) describes the flight-path angle at different altitudes $z$, and different values of $SC_D/m$, $SC_L/m$.

## 4. Concluding Remarks

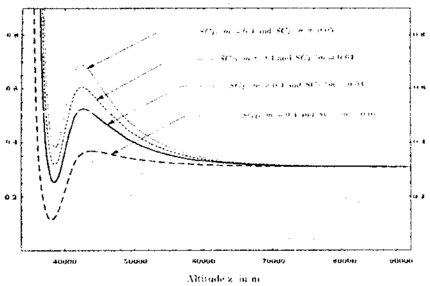
Though the method and theory developed in this paper are useful in their own right, they represent contributions to a broader context. In particular, the



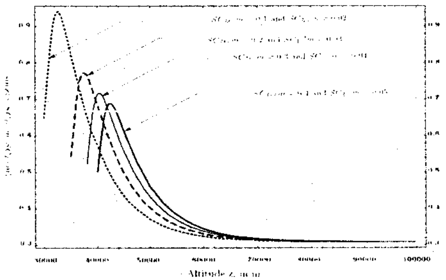
(a) The variation of $\sin\varphi(z, SC_D/m, SC_L/m)$ at constant value $SC_L/m = 0.01$ and different values for $SC_D/m$.



(b) The variation of $\sin\varphi(z, SC_D/m, SC_L/m)$ when $C_L/C_D = 0.1$.



(c) The variation of $\sin\varphi(z, SC_D/m. SC_L/m)$ at constant value $SC_D/m = 0.4$ and different values for $SC_L/m$.



(d) The variation of $\sin\varphi(z, SC_D/m, SC_L/m)$ at different values for $SC_D/m$ and $SC_L/m$.

Figure 1: The variation of $\sin\varphi(z, SC_D/m, SC_L/m)$ versus altitude $z$ and different values of $SC_D/m$ and $SC_L/m$.



solutions derived here can be used to perform an error analysis in velocity and flight path-angle during the maneuvers.

The equations which admits a closed-form solution expressed in terms of the Generalized Incomplete Beta function, the Exponential Integral, and the Generalized Exponential Integral proposed in Sections and , which are far from being simple, describes the atmospheric re-entry velocity and flight-path under very simple assumptions.

More precisely, the atmosphere density varies exponentially with the altitude, according to the Barometric Law, and the density at a specific altitude is practically constant around the Earth, i.e. the homogeneous atmosphere model is assumed.

Diurnal and latitudinal density variations, as well as space vehicle's centrifugal forces are neglected (see Duncan [10], pg.218). More complex effects can produce perturbations in the satellite trajectory. For example, the atmospheric drag due to the atmospheric rotation, or the oblateness of the Earth as well as the Moon-Sun attraction will change the dynamical equations of motion The gravitational acceleration remain constant and does not vary with the re-entry altitude $z$.

More recently Palacián [17] have analyzed perturbations in the orbits of a satellite due to inhomogeneous gravitational field. The space vehicle motion is planar, i.e., a 2-D motion.

# Annex

**Definition 1.** *The Generalized Incomplete Beta Function is defined by:*

$$\mathbb{B}(z_0, z_1, a, b) = \int_{z_0}^{z_1} x^{a-1}(1-x)^{b-1}dx \tag{B.1}$$

*where $z_0$ and $z_1$, are non-zero real numbers, or complex numbers with non-zero real real part, and $a, b \in \mathbb{R}$ such that $a \neq 1$, $b \neq 1$.*

**Theorem 1.** *If $\mathbb{B}(z_0, z_1, a, b)$ denote the Generalized Incomplete Beta Function,* *then the following evaluation holds:*

$$
\begin{aligned}
I_{1-p}(t) &= \int_{t_i}^{t} \cosh^{1-p}(At+\kappa)dt \\
&= \frac{1}{2^{2-p}A}\,\mathrm{Re}\left\{(-1)^{\frac{3-p}{2}}\mathbb{B}\left(z_0, z_1, 2-p, \frac{p-1}{2}\right)\right\} \\
I_{-1-p}(t) &= \int_{t_i}^{t} \cosh^{-1-p}(At+\kappa)dt \\
&= \frac{1}{2^{-p}A}\,\mathrm{Re}\left\{(-1)^{\frac{1-p}{2}}\mathbb{B}\left(z_0, z_1, -p, \frac{p+1}{2}\right)\right\}
\end{aligned}
\tag{B.2}
$$

*where*

$$
z_0 = 1+e^{2(At_i+\kappa)}, \quad z_1 = 1+e^{2(At+\kappa)} \tag{B.3}
$$

*and $p\in\mathbb{R}_+$. is any non-zero real positive number.*

**Proof** With the notation $x = At+\kappa$ using the relation $\cosh x = (e^{x}+e^{-x})/2 = (e^{2x}+1)/2e^{x}$, the first equality of Eq.(B.2) becomes:

$$
\begin{aligned}
I_{1-p}(t) &= \frac{1}{2^{1-p}}\int_{t_i}^{t}\frac{\left[1+e^{2(At+\kappa)}\right]}{e^{(At+\kappa)(1-p)}}dt \\
&= \frac{1}{2^{1-p}A}\int_{e^{At_i+\kappa}}^{e^{At+\kappa}}(u^2+1)^{1-p}u^{p-2}du
\end{aligned}
\tag{B.4}
$$

where a change of variable $u = e^{At+\kappa}$ is performed in the second equality in the Eq.(B.4). With a new change of variable $u^2+1 = x$ the Eq.(B.4) is transformed to:

$$
\begin{aligned}
I_{1-p}(t) &= \frac{(-1)^{\frac{3-p}{2}}}{2^{2-p}A}\int_{e^{2(At_i+\kappa)}+1}^{e^{2(At+\kappa)}+1} x^{1-p}(1-x)^{\frac{p-3}{2}}dx \\
&= \frac{1}{2^{2-p}A}\,\mathrm{Re}\left\{(-1)^{\frac{3-p}{2}}\mathbb{B}\left(z_0, z_1, 2-p, \frac{p-1}{2}\right)\right\}
\end{aligned}
\tag{B.5}
$$

where the integral in Eq.(B.5) is expressed with the Generalized Incomplete



Beta Function Eq.(1):

$$
\mathbb{B}\left(1+e^{2(At_i+\kappa)}, 1+e^{2(At+\kappa)}, 2-p, \frac{p-1}{2}\right) = \int_{e^{2(At_i+\kappa)}+1}^{e^{2(At+\kappa)}+1} x^{1-p}(1-x)^{\frac{p-3}{2}}dx \tag{B.6}
$$

With the notations for $z_0$ and $z_1$ defined by Eq.(B.3), after taking the real part of Cauchy Principal Value the first equality in Eq.(B.2) is demonstrated. Using now the integral for $I_{1-p}(t)$ from Eq.(B.2) then:

$$
\begin{aligned}
I_{-1-p}(t) &= \int_{t_i}^{t}\cosh^{-1-p}(At+\kappa)dt \\
&= \frac{1}{2^{-p}A}\,\mathrm{Re}\left\{(-1)^{\frac{1-p}{2}}\mathbb{B}\left(z_0, z_1, -p, \frac{p+1}{2}\right)\right\}
\end{aligned}
\tag{B.7}
$$

By combining Theorem (1) with the definition for $A$, $\kappa$ and $p$, introduced in the Eq.(33), this yields to Eq.(34).

**Definition 2.** *The Exponential Integral Function, denoted by* $\mathbf{Ei}(x)$, *is defined as:*

$$
\mathbf{Ei}(x) = \begin{cases} \int_{-\infty}^{x}\frac{e^t}{t}dt, & \forall x<0, \\ -\lim_{\varepsilon\to 0^+}\left[\int_{-x}^{-\varepsilon}\frac{e^{-t}}{t}dt + \int_{\varepsilon}^{\infty}\frac{e^{-t}}{t}dt\right], & \forall x>0. \end{cases} \tag{B.8}
$$

*where the case $x>0$, is an extension of the case $x<0$ obtained by taking the Cauchy Principal Value.*
*The Generalized Exponential Integral Function, denoted by* $\mathbf{E}_p(x)$, *is defined by the integral:*

$$
\mathbf{E}_p(x) = \int_{1}^{\infty}\frac{e^{-xt}}{t^p}dt \tag{B.9}
$$

*where $p\in\mathbb{R}$, $p\neq 0$, is a real non-zero constant.*

**Theorem 2.** *If* **Ei** *is the Exponential Integral Function defined for all $x$, then:*

$$
\int_{a}^{b}\frac{e^t}{t}dt = \mathbf{Ei}(b)-\mathbf{Ei}(a), \quad \forall a,b>0, a,b\in\mathbb{R} \tag{B.10}
$$

**Proof** First in the case $x > 0$, via the Eq.(B.8) from Definition(2):

$$\begin{aligned}\mathbf{Ei}(b) &= -\lim_{\varepsilon\to 0^+}\left[\int_{-b}^{-\varepsilon}\frac{e^{-t}}{t}dt + \int_{\varepsilon}^{\infty}\frac{e^{-t}}{t}dt\right]\\ \mathbf{Ei}(a) &= -\lim_{\varepsilon\to 0^+}\left[\int_{-a}^{-\varepsilon}\frac{e^{-t}}{t}dt + \int_{\varepsilon}^{\infty}\frac{e^{-t}}{t}dt\right]\end{aligned} \tag{B.11}$$

then the left hand side of Eq.(B.10) becomes:

$$\begin{aligned}\mathbf{Ei}(b) - \mathbf{Ei}(a) &= -\lim_{\varepsilon\to 0^+}\left[\int_{-b}^{-\varepsilon}\frac{e^{-t}}{t}dt - \int_{-a}^{-\varepsilon}\frac{e^{-t}}{t}dt\right]\\ = -\lim_{\varepsilon\to 0^+}\left[\int_{-b}^{-\varepsilon}\frac{e^{-t}}{t}dt + \int_{-\varepsilon}^{-a}\frac{e^{-t}}{t}dt\right] &= -\lim_{\varepsilon\to 0^+}\int_{-b}^{-a}\frac{e^{-t}}{t}dt\\ &= \int_{-a}^{-b}\frac{e^{-t}}{t}dt = \int_{a}^{b}\frac{e^{x}}{x}dx\end{aligned} \tag{B.12}$$

where the last equality of Eq.(B.12) is obtained by a change of variable $-t = x$. In the case $x < 0$, applying the first part of Definition(2):

$$\begin{aligned}\mathbf{Ei}(b) - \mathbf{Ei}(a) &= \int_{-\infty}^{b}\frac{e^{t}}{t}dt - \int_{-\infty}^{a}\frac{e^{t}}{t}dt\\ &= \lim_{\varepsilon\to -\infty}\left[\int_{\varepsilon}^{b}\frac{e^{t}}{t}dt - \int_{\varepsilon}^{a}\frac{e^{t}}{t}dt\right]\\ &= \lim_{\varepsilon\to -\infty}\left[\int_{\varepsilon}^{b}\frac{e^{t}}{t}dt + \int_{a}^{\varepsilon}\frac{e^{t}}{t}dt\right] = \int_{a}^{b}\frac{e^{t}}{t}dt\end{aligned} \tag{B.13}$$

and from the Eq.(B.12)and Eq.(B.13) the proof is completed. In the next theorem are established two new identities regarding the integrals of **Ei** function.

**Theorem 3.** *Let* **Ei** *be the Exponential Integral Function, and* $\mathbf{E}_p$ *be the Generalized Exponential Integral Function, then the following equalities holds:*

$$I_{m,n} = \int_{a}^{b}\frac{e^{mx}}{x}\frac{dx}{n + \mathbf{Ei}(mx)} = \ln\left[\frac{n + \mathbf{Ei}(mb)}{n + \mathbf{Ei}(ma)}\right] \tag{B.14}$$



*for* $m \in \mathbb{R}$, $m > 0$, $a, b \in \mathbb{R}$, $a < b$, *and*

$$\begin{aligned}\int_{a}^{b} x^{-p}\,\mathbf{Ei}(Ax)dx = \frac{1}{(p-1)}&\left\{a^{1-p}\left[\mathbf{E}_p(-aA) + \mathbf{Ei}(Aa)\right]\right.\\ &\left.-b^{1-p}\left[\mathbf{E}_p(-bA) + \mathbf{Ei}(bA)\right]\right\}\end{aligned} \tag{B.15}$$

*for* $a, b \in \mathbb{R}$, $a < b$.

**Proof** To proof Eq.(B.14) it is enough to show that

$$\begin{aligned}\frac{d}{dx}\ln\left[n + \mathbf{Ei}(mx)\right] &= \frac{1}{n + \mathbf{Ei}(mx)}\frac{d}{dx}\mathbf{Ei}(mx)\\ &= \frac{e^{mx}}{x}\frac{1}{n + \mathbf{Ei}(mx)}\end{aligned} \tag{B.16}$$

for any $x \in \mathbb{R}$, $x > 0$. It is a straight forward calculation to show that for any $m \in \mathbb{R}$, and $m > 0$, $\frac{d}{dx}\mathbf{Ei}(mx) = \frac{e^{mx}}{x}$. Differentiate the Eq.(B.8)with respect to $x$, then:

$$\begin{aligned}\frac{d}{dx}\mathbf{Ei}(mx) &= \frac{d}{dx}\left\{-\lim_{\varepsilon\to 0^+}\left[\int_{-mx}^{-\varepsilon}\frac{e^{-t}}{t}dt + \int_{\varepsilon}^{\infty}\frac{e^{-t}}{t}dt\right]\right\}\\ &= -\lim_{\varepsilon\to 0^+}\left[\frac{d}{dx}\int_{-mx}^{-\varepsilon}\frac{e^{-t}}{t}dt + \frac{d}{dx}\int_{\varepsilon}^{\infty}\frac{e^{-t}}{t}dt\right]\\ &= -\lim_{\varepsilon\to 0^+}\left[\frac{d}{dx}\int_{-mx}^{-\varepsilon}\frac{e^{-t}}{t}dt\right]\\ &= -\lim_{\varepsilon\to 0^+}\left[\frac{e^{mx}}{x} + \int_{-mx}^{-\varepsilon}\frac{d}{dx}\left(\frac{e^{-t}}{t}\right)dt\right] = \frac{e^{mx}}{x}\end{aligned} \tag{B.17}$$

where in the last equality of the Eq.(B.17) is used the well know formula:

$$\frac{d}{dx}\int_{A(x)}^{B} f(x,t)\,dt = -f(A(x),t)\frac{dA(x)}{dx} + \int_{A(x)}^{B}\frac{df(x,t)}{dx}\,dt \tag{B.18}$$

where $A(x) = mx$, $B = \varepsilon$, $f(x,t) = \frac{e^{-t}}{t}$ and the functions $A(x)$, $f(x,t)$ are continuous with respect the variables, and $f(x,t)$ does not depend on $x$. By combining the equalities Eq.(B.16) and Eq.(B.17) the Eq.(B.14) is proofed. To show the equality Eq.(B.15) first use an integration by parts which will produce:

$$\begin{aligned}\int_a^b x^{-p}\,\mathbf{Ei}(Ax)dx &= \frac{1}{(1-p)}\int_a^b \frac{d}{dx}(x^{-p+1})\,\mathbf{Ei}(Ax)dx \\ &= \frac{[a^{1-p}\,\mathbf{Ei}(Aa) - b^{1-p}\,\mathbf{Ei}(Ab)]}{(p-1)} + \frac{1}{(1-p)}\int_a^b x^{-p}e^{Ax}dx\end{aligned} \tag{B.19}$$

and to complete the proof will be enough to show that

$$\int_a^b x^{-p}e^{Ax}dx = a^{1-p}\,\mathbf{E}_p(-Aa) - b^{1-p}\,\mathbf{E}_p(-Ab) \tag{B.20}$$

Via the Definition(2), Eq.(B.9):

$$\mathbf{E}_p(-Aa) = \int_1^\infty t^{-p}e^{Aat}dt,\ \ \mathbf{E}_p(-Ab) = \int_1^\infty t^{-p}e^{Abt}dt \tag{B.21}$$

A change of variables $at = x$, respectively $bt = x$ in Eq.(B.21) yields:

$$\begin{aligned}\mathbf{E}_p(-Aa) &= \frac{1}{a^{p-1}}\int_a^\infty x^{-p}e^{Ax}dx \\ \mathbf{E}_p(-Ab) &= \frac{1}{b^{p-1}}\int_b^\infty x^{-p}e^{Ax}dx\end{aligned} \tag{B.22}$$

by subtracting the equalities in Eq.(B.22) then:

$$\begin{aligned}&a^{1-p}\,\mathbf{E}_p(-Aa) - b^{1-p}\,\mathbf{E}_p(-Ab) \\ &= \int_a^\infty x^{-p}e^{Ax}dx - \int_b^\infty x^{-p}e^{Ax}dx = \int_a^b x^{-p}e^{Ax}dx\end{aligned} \tag{B.23}$$

and by combining Eq.(B.23) and Eq.(B.19) one obtain the equality Eq.(B.15). The case $x < 0$, can be treated in a similar manner.

## References


[1] REGAN, F. J., *Re-Entry Vehicle Dynamics*, AIAA Educational Series, AIAA Inc., Oct. 1984.





[2] PRESS, W. H., TEUKOLSKY, S. A., VETTERLING, W. T., AND FLANNERY, B. P., *Numerical Recipes in C, C++, and Fortran*, Cambridge University Press, 2002, on-line at http://www.nr.com/oldverswitcher.html.

[3] EHRICKE, K. A., *Space Flight, Dynamics*, Vol. 2, In Series: Principles of Guided Missile Design, D. Van Nostrand Company, Inc., New York, 1st ed., 1962.

[4] CHAPMAN, D. R., An Approximate Analytical Method for Studying Entry Into Planetary Atmospheres, NACA Report NACA–TN–58–4276, May 1958, on-line at http://naca.larc.nasa.gov/.

[5] HANKEY, W. L., *Re-Entry Aerodynamics*, AIAA Educational Series, AIAA Inc., Oct. 1988.

[6] REGAN, F. J. AND ANANDAKRISHNAN, S. M., *Dynamics of Atmospheric Re-Entry*, AIAA Educational Series, AIAA Inc., Oct. 1993.

[7] LOH, W. H. T., *Dynamics and Thermodynamics of Planetary Entry*, Prentice-Hall, 1st ed., 1963.

[8] DUNNING, R. S., Study of a Guidance Scheme using Approximate Solutions of Trajectory Equations to Control the Aerodynamic Skip Flight of a Re-Entry Vehicle, NASA Langley Research Center, NASA TN–D–1923,Tech. Rep., Aug. 1963, pp. 821–831, on-line at http://ntrs.nasa.gov/.

[9] DUNNING, R. S., The Orbital Mechanics of Flight Mechanics, NASA Langley Research Center, NASA SP-325,Tech. rep., Oct. 1973, pp.821–831, on-line at http://ntrs.nasa.gov/.

[10] DUNCAN, R. C., *Dynamics of Atmospheric Entry*, McGraw-Hill Series in Missile and Space Technology, 1st ed., 1962.

[11] ABRAMOWITZ, M. AND STEGUN, I., *Handbook of Mathematical Functions*, New York, Nov. 1970, on-line at http://www.nr.com/oldverswitcher.html.

[12] GRADSHTEYN, I. AND RYZHIK, I., *Tables of Integrals, Series, and Products/Enlarged Edition*, Academic Press,New York, 1980.

[13] MARINESCU, A. AND STAICU, S., *Elements of Space Flight Mechanics*, Editura Tehnica, Bucureşti, 1st ed., 1997, (in romanian).

[14] BATTIN, R. H., *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA Educational Series, AIAA Inc., Oct. 1987.

[15] KNOTHE, H., *Satellites and the Riemannian Geometry*, Celestial Mechanics 1 36-45(1969).

[16] MITITELU, G., *Analytical solutions for the equations of motion of a space vehicle during the atmospheric re-entry phase on a 2-D trajectory*, Celestial Mechanics and Dynamical Astronomy (2009) (103) 327342 .

[17] PALACIÁN, J. F., *Dynamics of a satellite orbiting a planet with an inhomogeneous gravitational field*, Celestial Mechanics and Dynamical Astronomy 98, 219-249, 2007.

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# EXPLICIT ANALYTICAL SOLUTIONS FOR THE AVERAGE RUN LENGTH OF CUSUM AND EWMA CHARTS

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**Abstract**

We use the Fredholm type integral equations method to derive explicit formulas for the Average Run Length (ARL) in some special cases. In particular, we derive a closed form representation for the ARL of Cumulative Sum (CUSUM) chart when the random observations have hyperexponential distribution. For Exponentially Weighed Moving Average (EWMA) chart we solve the corresponding ARL integral equation when the observations have the Laplace distribution. The explicit formulas obviously takes less computational time than the other methods, e.g. Monte Carlo simulation or numerical integration.

## 1. Introduction

Cumulative Sum (CUSUM) chart was first proposed by Page (1954) in quality control in order to detect a small shift in the mean of a production process as

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soon as it occurs, as an extension to Shewhart's (1931) charts.

In practice, CUSUM charts are widely used in statistical control, to detect changes in the characteristics of a stochastic system e.g., mean or variance, see Brodsky and Darkhovsky [1], or Basseville and Nikiforov [2] for an introduction to CUSUM charts and their applications.

The recursive equation for CUSUM chart designed to detect an increase in the mean of observed sequence of nonnegative independent and identically distributed (i.i.d.) random variables $\xi_n$, is defined as

$$X_n = (X_{n-1} + \xi_n - a)^+, \; n = 1, 2, \ldots, \quad X_0 = x, \tag{1}$$

where $y^+ = \max(0, y)$. Several cases which lead to this representation are presented in [1], [2], and [4]. Denote by

$$\tau_b = \inf\{k \geq 0 : X_k \geq b\} \tag{2}$$

the first exit time of a random sequence $X_n$ over the positive level $b$, with $b > x - a$ (otherwise $\tau_b = 1$).

Let $\mathbb{P}_x$ and $\mathbb{E}_x$, denotes the probability measure and the induced expectation corresponding to the initial value $X_0 = x \geq 0$.

The problem studied in here is to find the Average Run Length (ARL) of the CUSUM procedure defined as a function $j(x) = \mathbb{E}_x \tau_b$.

The Exponentially Weighted Moving Average (EWMA) control chart was first proposed by Roberts (1959) in quality control, in order to detect a shift in the mean of a process. We consider here the EWMA as an AR(1), i.e., an Autoregressive Process of order one which is a simple generalization of a random walk (see [10]). We consider the AR(1) process described by the equation

$$X_t = \rho X_{t-1} + \eta_t \tag{3}$$

where $\rho \in (0, 1)$ and $\{\eta_t\}_{t \geq 1}$ is a sequence of independent identically distributed random variables, with $X_0 = x$. As a particularly case of the AR(1) one can obtain the EWMA chart in a standard form, by setting in Eq. (3), $\rho = 1 - \lambda$ and $\eta_t = \lambda \varepsilon_t$, with $\lambda \in (0, 1)$.

The problem is to find the expectation of the stopping time

$$\nu_b = \inf\{t \geq 0 : X_t \geq b\}, \; b > x. \tag{4}$$

One could use Monte Carlo simulation or numerical integration for both charts CUSUM and EWMA to find the ARL, but it is always desirable to have a closed form analytical solution to check the accuracy of the results.

The paper is organized as follow. In Section we obtain the ARL for CUSUM chart in the case of hyperexponential distribution (see Theorem 2.1).

It is well-known that any completely monotone probability density function can be approximated, by hyperexponential distributions, sometime also called mixture of exponentials. For example, Pareto and Weibull are completely monotone distributions, and so they can be approximated by mixture of exponentials (e.g., see [7] and [8]). Therefore, one can use the closed form representation given in Theorem 2.1 as an approximation for the cases when the random variables $\xi_n$ in Eq. (1) have Pareto or Weibull distributions.

In Section we discuss a closed form representation for $\mathbb{E}_x \nu_b$ and present the results in Theorem 2.2 for the case when the random variables $\eta_t$ in Eq. (3) has Laplace distribution. Our result generalize the result of Larralde [9], who use a different technique and obtain $\mathbb{E}_x \nu_b$ only in the particularly case $x = 0$ and $b = 0$. In Section we present several numerical examples.



## 2. The ARL Integral Equations for CUSUM and EWMA Procedures

It can be shown, see [5] and [6], that the ARL of the CUSUM chart, $j(x) = \mathbb{E}_x \tau_b$, is a solution of the integral equation

$$j(x) = 1 + \mathbb{E}_x\{I(0 < X_1 < b) j(X_1)\} + \mathbb{P}_x\{X_1 = 0\} j(0). \tag{5}$$

If $\xi_n$ are continuous distributed i.i.d random variables with a given d.f. $F(x)$, and density $f(x) = \frac{dF(x)}{dx}$, then we can write equation Eq. (5) as a Fredholm-type integral equation of the form

$$j(x) = 1 + j(0) F(a - x) + \int_0^b j(y) f(y + a - x) dy. \tag{6}$$

When $\xi_n$ are continuous i.i.d random variables with exponential distribution, then $F(x) = 1 - \exp(-x), x \geq 0$, Eq. (6) becomes

$$j(x) = 1 + \int_0^b j(y) e^{x - a - y} dy + (1 - e^{-(a-x)^+}) j(0). \tag{7}$$

and it's solution for $x \in [0, a]$ has the form (see e.g.[6])

$$j(x) = e^b (1 + e^a - b) - e^x, \quad \text{for} \quad x \in [0, a] \tag{8}$$

### 2.1 The ARL Fredholm Integral Equation for CUSUM chart with hyperexponential distributions

In this section we consider the case when the observed random variables $\xi_n$ have hyperexponential distribution with the d.f.

$$F(x) = 1 - \sum_{i=1}^{n} \lambda_i e^{-\alpha_i x} \tag{9}$$

with $\lambda_i \in \mathbb{R}_+$ subject to the condition that $\sum_{i=1}^{n} \lambda_i e^{-\alpha_i x}$ is a distribution function on $\mathbb{R}_+$, that is $\sum_{i=1}^{n} \lambda_i = 1$. Now, the Fredholm integral equation Eq. (6) can be written as follows

$$j(x) = 1 + \int_0^b j(y) \sum_{i=1}^{n} \lambda_i \alpha_i e^{\alpha_i (x-a-y)} dy + (1 - \sum_{i=1}^{n} \lambda_i e^{-\alpha_i (a-x)}) j(0), \quad 0 < x < a \tag{10}$$

**Theorem 1.1** *The solution of the integral equation* Eq. 10 *is*

$$j(x) = 1 + j(0) + \sum_{i=1}^{n} [d_i - \lambda_i j(0)] e^{\alpha_i (x-a)}, \quad for \quad x \in [0,a], \ b < a \tag{11}$$

where

$$j(0) = \frac{1 + \sum_{i=1}^{n} d_i e^{-\alpha_i a}}{\sum_{i=1}^{n} \lambda_i e^{-\alpha_i a}} \tag{12}$$

and the coefficients $d_i$, $i = 1, \dots, n$, are solutions of the linear system

$$\Delta \, \mathbf{d} = \mathbf{M} \tag{13}$$

where

$$\mathbf{d} = [d_1, d_2, \dots, d_n]^T,$$

$\Delta$ is the the non-singular matrix

$$\begin{bmatrix} D - M_{1.n} e^{-\alpha_1 a} - \lambda_1 \alpha_1 b e^{-\alpha_1 a} D & -M_{1.n} e^{-\alpha_2 a} - \lambda_1 \alpha_1 e^{-\alpha_2 a} A_{2.1} D & \dots & -M_{1.n} e^{-\alpha_n a} - \lambda_1 \alpha_1 e^{-\alpha_n a} A_{n.1} D \\ -M_{2.n} e^{-\alpha_1 a} - \lambda_2 \alpha_2 e^{-\alpha_1 a} A_{1.2} D & D - M_{2.n} e^{-\alpha_2 a} - \lambda_2 \alpha_2 b e^{-\alpha_2 a} D & \dots & -M_{2.n} e^{-\alpha_n a} - \lambda_2 \alpha_2 e^{-\alpha_n a} A_{n.2} D \\ \vdots & \vdots & \vdots & \vdots \\ -M_{n.n} e^{-\alpha_1 a} - \lambda_n \alpha_n e^{-\alpha_1 a} A_{1.n} D & -M_{n.n} e^{-\alpha_n a} - \lambda_n \alpha_n e^{-\alpha_2 a} A_{2.n} D & \dots & D - M_{n.n} e^{-\alpha_n a} - \lambda_n \alpha_n b e^{-\alpha_n a} D \end{bmatrix} \tag{14}$$

and

$$\mathbf{M} = \left[\lambda_1 (1 - e^{-b\alpha_1}) D + M_{1,n}, \lambda_2 (1 - e^{-b\alpha_2}) D + M_{2,n}, \dots \lambda_n (1 - e^{-b\alpha_n}) D + M_{n,n}\right]^T$$

with $D = \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a}$,

$$M_{k,n} = \left(1 - e^{-b\alpha_k}\right) \lambda_k - \lambda_k \alpha_k \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} A_{i,k} \tag{15}$$



and

$$A_{i,k} = \begin{cases} \frac{e^{(\alpha_i - \alpha_k) b} - 1}{(\alpha_i - \alpha_k)}, & i \neq k \\ b, & i = k \end{cases}$$

**Proof.** For $0 \le x \le a$ and $b < a$ Eq. (10) can be written as

$$j(x) = 1 + \sum_{i=1}^{n} d_i e^{\alpha_i (x-a)} + (1 - \sum_{i=1}^{n} \lambda_i e^{-\alpha_i (a-x)}) j(0), \quad 0 \le x \le a \tag{16}$$

where

$$d_i = \int_0^b j(y) \lambda_i \alpha_i e^{-\alpha_i y} dy, \quad i = 1, \dots, n. \tag{17}$$

or

$$j(x) = 1 + j(0) + \sum_{i=1}^{n} (d_i - \lambda_i j(0)) e^{\alpha_i (x-a)}, \quad 0 \le x \le a. \tag{18}$$

From Eq. (16) at $x = 0$ we obtain $j(0)$ given by Eq. (12)

To evaluate the coefficients $d_k$ for $k = 1, 2, \dots, n$ we substitute Eq. (18) in Eq. (17) and obtain

$$d_k = [1 + j(0)] \lambda_k \left(1 - e^{-b\alpha_k}\right) + \lambda_k \alpha_k \sum_{i=1}^{n} d_i e^{-\alpha_i a} A_{i.k} - j(0) \lambda_k \alpha_k \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} A_{i,k} \tag{19}$$

for $k = 1, 2, \dots, n$, where

$$A_{i.k} = \int_0^b e^{(\alpha_i - \alpha_k) y} dy = \begin{cases} \frac{e^{(\alpha_i - \alpha_k) b} - 1}{(\alpha_i - \alpha_k)}, & i \neq k \\ b, & i = k \end{cases}$$

Inserting the expression for $j(0)$ given by Eq. (16) in Eq. (19) we obtain a linear system of $n$ equations with $d_k$ unknowns $k = 1, 2, \dots, n$:

$$\begin{aligned} d_k \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} = \lambda_k \left(1 - e^{-b\alpha_k}\right) \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} + \left(1 + \sum_{i=1}^{n} d_i e^{-\alpha_i a}\right) \Bigg[ \left(1 - e^{-b\alpha_k}\right) \lambda_k \\ - \lambda_k \alpha_k \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} A_{i.k} \Bigg] \\ + \lambda_k \alpha_k \left(\sum_{i=1}^{n} \lambda_i e^{-\alpha_i a}\right) \left(\sum_{i=1}^{n} d_i e^{-\alpha_i a} A_{i.k}\right) = \lambda_k \left(1 - e^{-b\alpha_k}\right) \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} \\ + \left(1 + \sum_{i=1}^{n} d_i e^{-\alpha_i a}\right) M_{k.n} + \lambda_k \alpha_k \left(\sum_{i=1}^{n} \lambda_i e^{-\alpha_i a}\right) \left(\sum_{i=1}^{n} d_i e^{-\alpha_i a} A_{i.k}\right) \end{aligned}$$

$$k = 1, 2, \dots, n$$

where

$$M_{k,n} = \left(1 - e^{-b\alpha_k}\right)\lambda_k - \lambda_k\alpha_k \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} A_{i,k}$$

or

$$\begin{cases} d_1\left(D - M_{1,n}e^{-\alpha_1 a} - \lambda_1\alpha_1 b e^{-\alpha_1 a}D\right) + d_2\left(-M_{1,n}e^{-\alpha_2 a} - \lambda_1\alpha_1 e^{-\alpha_2 a}A_{2,1}D\right) + \dots \\ +d_n\left(-M_{1,n}e^{-\alpha_n a} - \lambda_1\alpha_1 e^{-\alpha_n a}A_{n,1}D\right) = \lambda_1\left(1 - e^{-b\alpha_1}\right)D + M_{1,n} \\ d_1\left(-M_{2,n}e^{-\alpha_1 a} - \lambda_2\alpha_2 e^{-\alpha_1 a}A_{1,2}D\right) + d_2\left(D - M_{2,n}e^{-\alpha_2 a} - \lambda_2\alpha_2 b e^{-\alpha_2 a}D\right) + \dots \\ +d_n\left(-M_{2,n}e^{-\alpha_n a} - \lambda_2\alpha_2 e^{-\alpha_n a}A_{n,2}D\right) = \lambda_2\left(1 - e^{-b\alpha_2}\right)D + M_{2,n} \\ \vdots \\ d_1\left(-M_{n,n}e^{-\alpha_1 a} - \lambda_n\alpha_n e^{-\alpha_1 a}A_{1,n}D\right) + d_2\left(-M_{n,n}e^{-\alpha_n a} - \lambda_n\alpha_n e^{-\alpha_2 a}A_{2,n}D\right) + \dots \\ +d_n\left(D - M_{n,n}e^{-\alpha_n a} - \lambda_n\alpha_n b e^{-\alpha_n a}D\right) = \lambda_n\left(1 - e^{-b\alpha_n}\right)D + M_{n,n} \end{cases} \tag{20}$$

with $D = \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a}$, and the proof of Theorem 1.1 is completed. □

## 2.2 Solution for the ARL Fredholm Integral Equation of EWMA chart with symmetric Laplace distributions

We analyze now the case of EWMA chart (see Eq. (3)) where $\eta_t \sim$ Laplace(0, 1). Recall that the density function of $\eta_t$ is given by $f(x) = \frac{1}{2}e^{-|x|}$.

It is well known that (e.g., see [5]) the function $h(x) = \mathbb{E}_x\nu_b$ is a solution of the following integral equation

$$h(x) = 1 + \mathbb{E}_x\left[I\{X_1 \le b\}\, h(X_1)\right]. \tag{21}$$

Then it can be shown that Eq. (21) becomes the following integral equation

$$h(x) = 1 + \frac{1}{2}\int_{\rho x}^{b} h(u)e^{\rho x - u}du + \frac{1}{2}\int_{0}^{+\infty} h(\rho x - y)e^{-y}dy. \tag{22}$$

The main result in this section is the following

**Theorem 2.1** *For any $0 < \rho < 1$ and $0 \le x < b$ the solution of the integral equation* Eq. (22) *is*

$$\begin{aligned} h(x) = E_x\nu_b = 2e^b - c_1(1+b) - \rho^2\left[2e^b - \left(1 + b + \frac{b^2}{2}\right)\right] \\ - c_1 e^b \sum_{k=3,5,7\dots}^{\infty} \rho^{k-1}P_1(k)\left[\frac{\Gamma(k+1,b)}{k!}\right] - e^b \sum_{k=4,6,8\dots}^{\infty} \rho^k P_2(k)\left[2 - \frac{\Gamma(k+1,b)}{k!}\right] \\ + c_1\left[x + \sum_{k=3,5,7\dots}^{\infty} \rho^{k-1}P_1(k)\frac{x^k}{k!}\right] + \left[-\rho^2\frac{x^2}{2!} - \sum_{k=4,6,8\dots}^{\infty} \rho^k P_2(k)\frac{x^k}{k!}\right] \end{aligned} \tag{23}$$



with the constant $c_1$ given by

$$c_1 = \frac{\rho(1-\rho^2) - \sum_{k=4,6,8,\dots}^{\infty} \rho^{k+1}P_2(k)}{(\rho - 1) + \sum_{k=3,5,7,\dots}^{\infty} \rho^k P_1(k)} \tag{24}$$

where $\Gamma(a,z) = \int_z^{+\infty} t^{a-1}e^{-t}dt$ denotes the Incomplete Gamma function, and

$$P_1(k) = \prod_{m=1}^{\left(\frac{k-1}{2}\right)} \left(1 - \rho^{2m-1}\right), \qquad P_2(k) = \prod_{m=2}^{\left(\frac{k}{2}\right)} \left(1 - \rho^{2m-2}\right) \tag{25}$$

**Proof.** It can be shown that Eq. (22) can be reduced to the following second order differential equation

$$h''(x) = \rho^2 h(x) - \rho^2 h(\rho x) - \rho^2 \tag{26}$$

We try to find a series solution for Eq. (26) of the form

$$h(x) = \sum_{k=0}^{\infty} \frac{c_k x^k}{k!} = c_0 + \sum_{k=1}^{\infty} \frac{c_k x^k}{k!} \tag{27}$$

Then from Eq. (27) $j(0) = c_0$ and from Eq. (26) at $x = 0$, $h''(0) = -\rho^2$. It can be shown also that the coefficients $c_k$ satisfied the non-linear recurrent equation

$$c_{k+2} = \rho^2(1 - \rho^k)c_k \quad \text{for} \quad k \ge 1 \tag{28}$$

and using the recurrence Eq. (28) we may find that

$$\begin{aligned} c_k &= -\rho^k \prod_{m=2}^{\left(\frac{k}{2}\right)} \left(1 - \rho^{2m-2}\right) \quad \text{for } k = 4, 6, 8, \dots \\ c_k &= c_1\rho^{k-1} \prod_{m=1}^{\left(\frac{k-1}{2}\right)} \left(1 - \rho^{2m-1}\right) \quad \text{for } k = 3, 5, 7, \dots \end{aligned} \tag{29}$$

With the coefficients $c_1$ and $c_0$ given by

$$c_1 = \frac{\rho(1-\rho^2) - \sum_{k=4,6,8,\dots}^{\infty} \rho^{k+1}P_2(k)}{(\rho - 1) + \sum_{k=3,5,7,\dots}^{\infty} \rho^k P_1(k)} \quad \text{with } 0 < \rho < 1 \tag{30}$$

$$c_0 = 2e^b - c_1(1+b) - \rho^2\left[2e^b - \left(1+b+\frac{b^2}{2}\right)\right] - c_1 e^b \sum_{k=3,5,7\ldots}^{\infty} \rho^{k-1}P_1(k)\left[\frac{\Gamma(k+1,b)}{k!}\right] - e^b \sum_{k=4,6,8,\ldots}^{\infty} \rho^k P_2(k)\left[2 - \frac{\Gamma(k+1,b)}{k!}\right] \quad (31)$$

The solution for the integral equation Eq. (22) is

$$h(x) = E_x\nu_b = c_0 + c_1\left[x + \sum_{k=3,5,7\ldots}^{\infty} \rho^{k-1}P_1(k)\frac{x^k}{k!}\right] + \left[-\rho^2\frac{x^2}{2!} - \sum_{k=4,6,8\ldots}^{\infty} \rho^k P_2(k)\frac{x^k}{k!}\right] \quad (32)$$

where the constant $c_1$ is given by Eq. (30), and the proof of Theorem 2.1 is completed.

## 3. Comparisons of the results with numerical integration and Monte Carlo simulations

In this section we present the scheme to evaluate numerically the solutions of the integral equation Eq. (10) (see also Eq. (6)).

By elementary quadrature rule we can approximate, in general, the integral $\int_0^b f(y)dy$ by a sum of areas of rectangles with bases $b/m$ with heights chosen as the values of $f$ at the midpoints of intervals of length $b/m$ beginning at zero.i.e. on the interval $[0, b]$ with the division points $0 \le a_1 \le a_2 \le \ldots \le a_m \le b$ and weights $w_k = b/m \ge 0$, we can writing

$$\int_0^b f(y)dy \approx \sum_{k=1}^{m} w_k f(a_k) \quad \text{with} \quad a_k = \frac{b}{m}\left(k - \frac{1}{2}\right), \quad k = 1, 2, \ldots m.$$

If $j^*(x)$ denotes the approximated solution of $j(x)$ then the last term in the Eq. (6) can be expressed as

$$\sum_{k=1}^{m} w_k j^*(a_k) f(a_k + a - a_i), \quad i = 1, 2, \ldots m. \quad (33)$$

and the integral equation Eq. (6) becomes the following system of $m$ linear equations in the $m$ unknowns $j^*(a_1), j^*(a_2), \ldots, j^*(a_m)$

$$\begin{cases} j^*(a_1) = 1 + j^*(a_1)\left[F(a-a_1) + w_1 f(a)\right] + \sum\limits_{k=2}^{m} w_k j^*(a_k) f(a_k + a - a_1) \\ j^*(a_2) = 1 + j^*(a_1)\left[F(a-a_2) + w_1 f(a_1 + a - a_2)\right] + \sum\limits_{k=2}^{m} w_k j^*(a_k) f(a_k + a - a_2) \\ \vdots \\ j^*(a_m) = 1 + j^*(a_1)\left[F(a-a_m) + w_1 f(a_1 + a - a_m)\right] + \sum\limits_{k=2}^{m} w_k j^*(a_k) f(a_k + a - a_m) \end{cases} \quad (34)$$



For numerical implementation is preferable to writing the linear system Eq. (34) in matrix form as

$$\mathbf{J}_{m\times 1} = \mathbf{1}_{m\times 1} + \mathbf{R}_{m\times m}\mathbf{J}_{m\times 1} \quad \text{or} \quad (\mathbf{I}_m - \mathbf{R}_{m\times m})\mathbf{J}_{m\times 1} = \mathbf{1}_{m\times 1} \quad (35)$$

where

$$\mathbf{J}_{m\times 1} = \begin{pmatrix} j^*(a_1) \\ j^*(a_2) \\ \vdots \\ j^*(a_m) \end{pmatrix}, \mathbf{1}_{m\times 1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad (36)$$

$$\mathbf{R}_{m\times m} = \begin{pmatrix} F(a-a_1) + w_1 f(a) & w_2 f(a_2 + a - a_1) & \ldots & w_m f(a_m + a - a_1) \\ F(a-a_1) + w_1 f(a_1 + a - a_2) & w_2 f(a) & \ldots & w_m f(a_m + a - a_2) \\ \vdots & \vdots & & \vdots \\ F(a-a_m) + w_1 f(a_1 + a - a_m) & w_2 f(a_2 + a - a_m) & \ldots & w_m f(a) \end{pmatrix} \quad (37)$$

and $\mathbf{I}_m = \text{diag}(1, 1, \ldots, 1)$ is the unit matrix of order $m$. If there exists $(\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}$, then the solution of the matrix equation Eq. (35) is

$$\mathbf{J}_{m\times 1} = (\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times 1}$$

Solving this set of equations for the approximate values of $j^*(a_1), j^*(a_2), \ldots, j^*(a_m)$, we may approximate the function $j(x)$ as

$$j(x) \approx 1 + j^*(a_1)F(a-x) + \sum_{k=1}^{m} w_k j^*(a_k) f(a_k + a - x) \text{ with } w_k = \frac{b}{m} \text{ and } a_k = \frac{b}{m}\left(k - \frac{1}{2}\right) \quad (38)$$

**Numerical Examples** We denotes by $j^*(x)$ the approximated solution, respectively by $j(x)$ the exact solutions, and define here the relative errors as $\epsilon_r = \mid j(x) - j^*(x) \mid / j(x)$.

For some values of $a$, $b$ and the number of divisions $m$ specified several numerical examples are presented in Table 1 and Table 2.

**Mixture of two exponentials**

For mixture of two exponentials the integral equation Eq. (6) is

$$j(x) = 1 + \int_0^b j(y)\sum_{i=1}^{2}\lambda_i\alpha_i e^{\alpha_i(x-a-y)}dy + \left(1 - \sum_{i=1}^{2}\lambda_i e^{-\alpha_i(a-x)}\right)j(0) \quad (39)$$

The coefficients $M_{k,n}$ for $n = 2$ and $k = 1, 2$ are

$$\begin{aligned} M_{1,2} &= \lambda_1\left(1 - e^{-b\alpha_1}\right) - \lambda_1\alpha_1 \sum_{i=1}^{2} \lambda_i e^{-\alpha_i a} A_{i,1} \\ &= \lambda_1\left(1 - e^{-b\alpha_1}\right) - \lambda_1\alpha_1 \left[\lambda_1 e^{-\alpha_1 a} b + \lambda_2 e^{-\alpha_2 a}\left(\frac{e^{(\alpha_2-\alpha_1)b} - 1}{\alpha_2 - \alpha_1}\right)\right] \\ M_{2,2} &= \lambda_2\left(1 - e^{-b\alpha_2}\right) - \lambda_2\alpha_2 \sum_{i=1}^{2} \lambda_i e^{-\alpha_i a} A_{i,2} \\ &= \lambda_2\left(1 - e^{-b\alpha_2}\right) - \lambda_2\alpha_2 \left[\lambda_1 e^{-\alpha_1 a}\left(\frac{e^{(\alpha_1-\alpha_2)b} - 1}{\alpha_1 - \alpha_2}\right) + \lambda_2 e^{-\alpha_2 a} b\right] \end{aligned} \quad (40)$$

with

$$A_{1,1} = A_{2,2} = b \text{ and } A_{1,2} = \left(\frac{e^{(\alpha_1-\alpha_2)b} - 1}{\alpha_1 - \alpha_2}\right), \quad A_{2,1} = \left(\frac{e^{(\alpha_2-\alpha_1)b} - 1}{\alpha_2 - \alpha_1}\right)$$

The solution is

$$j(x) = 1 + \sum_{i=1}^{2} d_i e^{\alpha_i(x-a)} + \left(1 - \sum_{i=1}^{2} \lambda_i e^{-\alpha_i(a-x)}\right) j(0) \quad (41)$$

with

$$j(0) = \frac{\left(1 + \sum_{i=1}^{2} d_i e^{-\alpha_i a}\right)}{\sum_{i=1}^{2} \lambda_i e^{-\alpha_i a}} \quad (42)$$

and the coefficients $d_1$, and $d_2$ are obtained as the solutions of the linear system Eq. (13).

$$\begin{cases} d_1\left[D - M_{1,2}e^{-\alpha_1 a} - \lambda_1\alpha_1 b e^{-\alpha_1 a} D\right] + d_2\left[-M_{1,2}e^{-\alpha_2 a} - \lambda_1\alpha_1 e^{-\alpha_2 a} A_{2,1} D\right] \\ = \lambda_1\left(1 - e^{-b\alpha_1}\right) D + M_{1,2} \\ \\ d_1\left[-M_{2,2}e^{-\alpha_1 a} - \lambda_2\alpha_2 e^{-\alpha_1 a} A_{1,2} D\right] + d_2\left[D - M_{2,2}e^{-\alpha_2 a} - \lambda_2\alpha_2 b e^{-\alpha_2 a} D\right] \\ = \lambda_2\left(1 - e^{-b\alpha_2}\right) D + M_{2,2} \end{cases} \quad (43)$$

with $D = \sum_{i=1}^{2} \lambda_i e^{-\alpha_i a}$.

Numerical results are presented in Table 1, for the case $m = 800$ division points.

**Mixture of four exponentials**



Table 1: **Comparisons with the numerical results for mixture of two exponentials**

| $x = 0$ | | $\lambda_1 = \lambda_2 = 0.5$, $\alpha_1 = 1.5$, $\alpha_2 = 2.8$, | | | $\lambda_1 = 0.3$, $\lambda_2 = 0.7$, $\alpha_1 = 1.1$, $\alpha_2 = 3.5$ | | |
|---|---|---|---|---|---|---|---|
| a | b | $j(x)$ | $j^*(x)$ | $\epsilon_r(\%)$ | $j(x)$ | $j^*(x)$ | $\epsilon_r(\%)$ |
| 2.5 | 0.5 | 175.965 | 175.745 | 0.12 | 89.995 | 89.914 | 0.09 |
| 3.0 | 1.0 | 799.111 | 797.115 | 0.24 | 270.156 | 269.666 | 0.18 |
| 3.5 | 1.5 | 3597.65 | 3584.14 | 0.37 | 811.241 | 809.023 | 0.27 |
| 4.0 | 2.0 | 16158.2 | 16076.3 | 0.5 | 2438.48 | 2429.55 | 0.36 |
| 4.5 | 2.5 | 72504.7 | 72033.5 | 0.64 | 7332.76 | 7298.95 | 0.46 |
| 5.0 | 3.0 | 325183 | 325095 | 0.02 | 22050.8 | 21927.1 | 0.56 |
| 5.5 | 3.5 | $1.45801 \times 10^6$ | $1.45782 \times 10^6$ | 0.01 | 66299.6 | 658551 | 0.67 |

In the case of mixture of four exponentials, the integral equation Eq. (6) becomes

$$j(x) = 1 + \int_0^b j(y) \sum_{i=1}^{4} \lambda_i\alpha_i e^{\alpha_i(x-a-y)} dy + \left(1 - \sum_{i=1}^{4} \lambda_i e^{-\alpha_i(a-x)}\right) j(0) \quad (44)$$

and the coefficients $M_{k,n}$ for $n = 4$ and $k = 1, 2, 3, 4$ given by Eq. (15) and $A_{i,i} = b$, $i = 1, \ldots, 4$.

Table 2: **Comparisons with the numerical results for mixture of four exponentials**

| $a = 2.3$, $b = 1.5$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1/3$, $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, $\alpha_3 = 1.1$, $\alpha_4 = 1.3$ | | | |
|---|---|---|---|
| x | $j(x)$ | $j^*(x)$ | $\epsilon_r(\%)$ |
| 0.0 | 15.614 | 15.594 | 0.129 |
| 0.5 | 15.240 | 15.221 | 0.128 |
| 1.0 | 14.702 | 14.683 | 0.127 |
| 1.5 | 13.912 | 13.895 | 0.125 |
| 2.0 | 12.729 | 12.714 | 0.121 |
| 2.5 | 10.914 | 10.902 | 0.113 |
| 3.0 | 8.061 | 8.053 | 0.092 |

The solution is

$$j(x) = 1 + \sum_{i=1}^{4} d_i e^{\alpha_i(x-a)} + \left(1 - \sum_{i=1}^{4} \lambda_i e^{-\alpha_i(a-x)}\right) j(0)$$

with $j(0) = \frac{\left(1+\sum_{i=1}^{4} d_i e^{-\alpha_i a}\right)}{\sum_{i=1}^{4} \lambda_i e^{-\alpha_i a}}$; $d_1$, $d_2$, $d_3$ and $d_4$ are the solutions of the linear algebraic system Eq. (13)

For the number of division points $m = 600$, we present in Table 2 several numerical results in the case of mixture of four exponentials. In Table 3 we present the results of Monte Carlo simulations for ARL when the EWMA chart is used with symmetric Laplace distributed random variables, and compare it with the closed-form expression given by the Eq. (23) in Theorem 2.1.

Table 3: $E_x\nu_b$ **for** $x = 0.3$ **and different** $b$ **and** $\rho$ **with** $b > x$. **Comparisons with MC simulations**

| $\rho$ | $b = 0.4$ | | $b = 0.6$ | | $b = 0.8$ | | $b = 1.0$ | |
|---|---|---|---|---|---|---|---|---|
| | ARL | MCsim | ARL | MCsim | ARL | MCsim | ARL | MCsim |
| 0.1 | 3.090 | 3.090 | 3.769 | 3.771 | 4.594 | 4.591 | 5.596 | 5.598 |
| 0.2 | 3.197 | 3.192 | 3.893 | 3.898 | 4.732 | 4.730 | 5.745 | 5.745 |
| 0.3 | 3.311 | 3.313 | 4.025 | 4.027 | 4.877 | 4.878 | 5.896 | 5.898 |
| 0.4 | 3.442 | 3.444 | 4.176 | 4.174 | 5.041 | 5.042 | 6.065 | 6.060 |
| 0.5 | 3.604 | 3.602 | 4.361 | 4.363 | 5.243 | 5.245 | 6.272 | 6.281 |
| 0.6 | 3.819 | 3.823 | 4.609 | 4.606 | 5.513 | 5.523 | 6.552 | 6.568 |
| 0.7 | 4.134 | 4.133 | 4.973 | 4.970 | 5.914 | 5.924 | 6.975 | 6.982 |
| 0.8 | 4.668 | 4.668 | 5.594 | 5.599 | 6.610 | 6.625 | 7.726 | 7.716 |
| 0.9 | 5.901 | 5.901 | 7.038 | 7.052 | 8.248 | 8.240 | 9.537 | 9.536 |

## 4. Conclusions

We have used the integral equations method to obtain closed form analytical expressions for the ARL of the CUSUM and EWMA control charts, when the observed random variables have hyperexponential distribution for CUSUM chart, respectively symmetric Laplace distribution for EWMA chart. We compare our analytical results with the numerical one and the Monte Carlo simulations. The methods are consistent with a high level of accuracy up to 98%.



## References


[1] B.E. Brodsky, B.S. Darkhovsky, "Nonparametric Change-Point Problems", Kluwer Academic Publisher, 1993.

[2] M. Basseville, I.V. Nikiforov, "Detection of Abrupt Changes-Theory and Applications", Prentice-Hall, Inc. Englewood Cliffs, N.J., available online at http://www.irisa.fr/sisthem/kniga/, 1993.

[3] George V. Moustakides, Aleksey S. Polunchenko and Alexander G. Tartakovsky, A Numerical Approach to Performance Analysis of Quickest Change-Point Detection Procedures, Statistica Sinica, 2009 (in print).

[4] Mazalov, V.V. and Zhuravlev, D.N., A method of Cumulative Sums in the problem of Detection of traffic in computer networks, Programming and Computer Software **28**(6) (2002) 342-348.

[5] Venkateshwara, B.R., Ralph, L.D., Joseph J.P., Uniqueness and convergence of solutions to average run length integral equations for cumulative sums and other control charts, IIE Transactions **33** (2001) 463-469.

[6] Vardeman, S. and Ray, D., Average Run Lengths for CUSUM schemes when observations are Exponentially Distributed, Technometrics **27**(2) (1985) 145-150.

[7] Feldmann, A. and Whitt, W., Fitting mixtures of exponentials to long-tail distributions to analyze network performance models , Performance Evaluation **31**(1998) 245-279.

[8] Dufresne, D., A Fitting combinations of exponentials to probability distributions, Applied Stochastic Models in Business and Industry **23**(1) (2006) 23-48.

[9] Larralde, L., A first passage time distribution for a discrete version of the Ornstein-Uhlenbeck process, Journal of Physics A: Mathematical and General **37** (2004) 3759-3767.

[10] Gustafsson, F., "Adaptive Filtering and Change Detection", John Wiley & Sons, 2000.


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