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Contributed Paper

Confidence Interval Estimation for Right-Tailed Deviation Risk Measures Under Heavy-Tailed Losses

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ABSTRACT

The estimation of the price of an insurance risk is a very important actuarial problem. This price has to reflect the property of the distribution of the random variable describing the corresponding loss. If the loss variable has a heavy-tailed distribution (i.e. distribution with an infinite variance) then, the risk measure (as a measure of the risk premium) should be higher. For providing risk measures with heavy-tailed distributions, standard procedures from classical statistics (when the variance is finite) cannot be applied. In this paper we propose confidence interval estimation for the Wang's right-tailed deviation risk measure for heavy-tailed losses.

Keywords: Heavy-tailed distribution, Wang's right-tailed deviation, risk measure, Hill estimator.

1. INTRODUCTION

A main actuarial problem involves providing the risk measure of an insurance risk. This risk measure has to reflect the property of the distribution of the random variable describing the corresponding loss. For example, when considering the variability of the loss variable, the shape of the distribution and particularly the tail behavior influence the risk measure. If the loss variable has a heavy-tailed distribution, then the price should be higher. In risk management, normal distributions have been commonly used to model those of the loss variables. However, the empirical studies conclude that financial and actuarial data exhibit systematic deviations from normality and usually have heavier tails than the Gaussian model. Therefore, it is appropriate to model the underlying distribution function F as one which is heavy-tailed. Recently, the empirical estimation of risk measures and related quantities were proposed by Jones and Zitikis [1]. Interval estimations of various actuarial risk measures: proportional hazards transform (PHT), Wang Transform (WT), Value-at-risk (VaR) and conditional tail expectation (CTE) were proposed by Kaiser and Brazauskas [2] for the case of finite variances. Nevertheless, for heavy-tailed losses, these results are not appropriate when the variance of the loss variable is infinite.

Many authors proposed the estimation of several actuarial risk measures for heavy-tailed distributions. For example, Necir and Meraghni [3] proposed the estimation of the proportional hazard premium for heavy-tailed claim amounts. Necir et al. [4] proposed the estimation of the conditional tail expectation for heavy-tailed losses.

In actuarial science and finance, traditionally, a common used measure of risks is the standard deviation. The standard deviation is a "standard" measure of the deviation from the mean of a distribution with finite variance. Even if the standard deviation has been used to measure the deviation from the mean for general distributions, it is not a good risk measure for large insurance risks with heavy-tailed distributions, as reported by many authors, for example, Ramsay [5] and Lowe and Stanard [6]. Wang [7] had proposed the right-tailed deviation as a new risk measure, which has the common ordering of risks such as first and second stochastic dominance. Moreover, the right-tailed deviation is additive when the risk is divided into excess of loss layers. Hürlimann [8] showed that Wang's right- tailed deviation is also a degree 2 tail free risk measure while conditional value at risk and Wang transform are not. Therefore, it is of interest to construct a new estimator of a risk measure by Wang's right-tailed deviation.

In this paper we propose the confidence interval estimation for the Wang's right-tailed deviation risk measure under heavy-tailed losses. We focus on the popular tail index estimator due to Hill [9]. Then we use the well-known high quantile estimator to estimate high quantile, $F^{-1}(1-s)$ for sufficiently small s, proposed by Weissman [10], since high quantile is better than empirical distribution when the underlying distribution is heavy-tailed, refer to Kim and Hardy [11], and Necir and Meraghni [3].

This paper is organized as follows: in Section 2 we introduce heavy-tailed distributions. We explain the Wang's right-tailed deviation in Section 3. In Section 4, we construct a risk measure estimation. We conclude in Section 5. In Section 6, we provide the proofs of our results.

2. HEAVY-TAILED DISTRIBUTIONS

In this section, we introduce heavy-tailed distributions. Heavy-tailed distributions are appropriate for modeling the random fluctuations of typical phenomena such as flood levels of rivers, insurance claims, high wind speeds, as well as situations in economics, ecological systems, internet traffic, finance and business. Examples of applications can be found in many book see Beirlant et al.[12], Markovich [13], and Resnick [14]). Examples of heavy-tailed distributions are Pareto, Cauchy, Log-gamma, Burr, t-distribution.

The distribution of X is said to have a heavy-tailed distribution when its tail probability $\bar{F}(x) := P(X>x)$ is regularly varying at infinity i.e., there exists some $\alpha > 0$ such that for all $t > 0$

$$\frac{\bar{F}(tx)}{\bar{F}(x)} \to t^{-\alpha} \tag{1}$$

as $x \to \infty$, where α be the index of regular variation. Equivalently,

$$\bar{F}(x) = x^{-\alpha} L(x),\ x > 0 \tag{2}$$

where $L(x)$ is a slowly varying function. Speaking roughly, heavy tailed distributions have tails that are thicker than the exponential distribution. The class of distributions satisfying (1) includes the distributions in the domain of attraction of stable laws (see Hill [15]).

A distribution function is in the domain of attraction of a stable law with stability index $0 < \alpha < 2$, notation: $F \in D(\alpha)$, if there are two real sequences $A_n > 0$ and C_n such that

$$A_n^{-1}\left(\sum_{i=1}^{n} X_i - C_n\right) \xrightarrow{D} S_\alpha(\sigma, \delta, \mu),$$

$$\text{as } n \to \infty \tag{3}$$

where D denotes convergence in distribution, and where $S_\alpha(\sigma,\delta,\mu)$ is a stable distribution with parameter $0 < \alpha \leq 2$, $-1 \leq \delta \leq 1$, $\sigma > 0$ and $-\infty \leq \mu \leq \infty$. In addition, if we denote by $G(x) := P(|X| \leq x) = F(x) - F(-x), x > 0$, the distribution function of $Z := |X|$, then the tail behavior of $F \in D(\alpha)$, for $0 < \alpha < 2$, is described as follows;

(i) The tail $1 - G$ is regularly varying at infinity with index α, i. e.

$$\lim_{t\to\infty} \frac{(1-G(xt))}{(1-G(t))} = x^{-\alpha}, x > 0 \qquad (4)$$

(ii) There exists $0 \leq p \leq 1$ such that

$$\lim_{x\to\infty} \frac{(1-F(x))}{(1-G(x))} = p,$$

$$\lim_{x\to\infty} \frac{F(-x)}{(1-G(x))} = 1 - p = q \qquad (5)$$

There are many ways to identify whether a distribution is heavy-tailed or not. One of the convenient graphical tools is the QQ-plot, where the quantiles of empirical distribution are plotted vs. the exponential distribution. If QQ-plot is linear then, the sample comes from a distribution with medium-sized tails. If the plot is concave then, the sample comes from a distribution with heavy tails, otherwise it is convex, and the sample comes from a distribution with short tails (see, Gencay et al. [16]).

The tail index has a important role in studying heavy-tailed distributions. It determines the shape of the tail. The tail index estimator is computed based on the proportion of extreme value statistics in the sample distribution. Hill estimator is a popular estimator which is proposed by Hill [9]. Hill estimator is a maximum likelihood estimator for the tail of a distribution.

$$\hat{\gamma} := \frac{1}{k}\sum_{j=1}^{k} \log^{+}\left(\frac{X_{(n-j+1)}}{X_{(n-k)}}\right) \qquad (6)$$

where $X_{(k)}$ denotes the kth-order statistics of a random sample $X_1, X_2, \ldots, X_n$ from X. We can rewrite as

$$\hat{\gamma} := \frac{1}{k}\sum_{j=1}^{k} \log^{+} X_{(n-j+1)} - \log^{+} X_{(n-k)} \qquad (7)$$

where $\log^{+} u := \max(0, \log u)$. This estimator is consistent for $\gamma = \frac{1}{\alpha}$, i.e.

for $\quad k_n \to \infty, \frac{k_n}{n} \to 0.$

We have

$$\hat{\gamma}_n \xrightarrow{P} \gamma$$

where P denotes convergence in probability.

Properties of Hill Estimator have been established by Mason [17] assuming only that the underlying distribution is regularly varying at infinity. Asymptotic normality of $\hat{\gamma}_n$ has been investigated under various conditions by a number of researchers (see de Hann and Resnick [18], Deheuvels et al. [19], and Berilant and Teugel [20]).

The right and left high quantiles of small t of distribution function F, respectively, are two reals x_R and x_L defined by $1 - F(x_R) = t$ and F (t, i.e. $x_R = F^{-1}(1-t)$ and $x_L = F^{-1}(t)$ Let $k = k_n$ and $l = l_n$ be sequences of integers (called trimming sequences) satisfying

$1 < k < n$, $1 < l < n$, $\frac{k}{n} \to 0$ and $\frac{l}{n} \to 0$, as $n \to \infty$. Weissman estimators of high quantiles x_L and x_R are defined by as

$$\hat{x}_L = F_L^{-1}(t) := \left(\frac{k}{n}\right)^{\gamma_L} X_{(k)} t^{-\gamma_L}, \text{as } t \to 0$$

$$\hat{x}_R = F_R^{-1}(1-t) := \left(\frac{l}{n}\right)^{1/\alpha_R} X_{(n-l)} t^{-1/\alpha_R},$$

as $t \to 0$ (8)

where

$$\frac{1}{\hat{\alpha}_L} = \hat{\gamma}_L = \hat{\gamma}_L(k)$$
$$:= \frac{1}{k}\sum_{i=1}^{k} \log^+(-X_{(i)}) - \log^+(-X_{(k)})$$
$$\frac{1}{\hat{\alpha}_R} = \hat{\gamma}_R = \hat{\gamma}_R(l)$$
$$:= \frac{1}{l}\sum_{i=1}^{l} \log^+(X_{(n-i+1)}) - \log^+(X_{(n-l)}) \qquad (9)$$

are two form of Hill estimator for tail index α which is also estimated, using the order statistics $z_{(1)} \leq z_{(2)} \leq \ldots z_{(n)}$ associated to a sample $(z_1, z_2, \ldots, z_n)$ from z, as follows:

$$\hat{\gamma} = \hat{\gamma}(m) := \frac{1}{m}\sum_{i=1}^{m} \log^+(Z_{(n-i+1)}) - \log^+(Z_{(n-m)}) \qquad (10)$$

where $m = m_n$ is an intermediate sequence the same conditions as k and l.

It is important to determine the optimal threshold k and l. A number of methods for estimation optimal and were proposed in the literature (see, Dekker and de Hann [21], Cheng and Peng [22], and Dees et al. [23])

3. THE WANG'S RIGHT-TAIL DEVIATION

Wang [7] has proposed the right-tailed deviation as a new risk measure when the underlying distribution is heavy-tailed, which is motivated from various perspectives such as the certainty equivalent approach, the parametric approach and distance between loss distribution. He showed that the right-tailed deviation is consistent with the first and second stochastic dominance, and is appropriate when a risk is divided into excess of layers, which is advantageous in calculating risk charges in reinsurance pricing. The Wang's right-tailed deviation risk measure is defined by

$$\rho(X) = \int_0^\infty \sqrt{1-F(x)}\,dx - E(X)$$
$$= \int_0^\infty \sqrt{1-F(x)}\,dx - \int_0^\infty (1-F(x))\,dx$$
$$= \int_0^1 (\psi_1(s) - \psi_2(s)) F^{-1}(s)\,ds$$
$$= \int_0^1 \psi(s) F^{-1}(s)\,ds \qquad (11)$$

where

$\psi_1(s) = \frac{1}{2\sqrt{1-s}}$, $\psi_2(s) = 1$, and $\psi(s) = \frac{1}{2\sqrt{1-s}} - 1.$

In this form, the Wang's right tail deviation risk measure is similar to a spectral risk measure (see Sriboonchitta et al. [24]).

Indeed, this function $\psi(s) = \frac{1}{2\sqrt{1-s}} - 1$ is not the weight function. But, nevertheless, we can use this function to construct estimation for right-tailed deviation risk measures by the L-Functional, proposed by Necir and Meraghni [25]. For application, we need the following regularity assumptions on function ψ are required

(A1) ψ is differentiable on (0,1).

(A2) $\lim_{s\to 0} \frac{\psi(1-s)}{\psi(s)} = 2 < \infty$

$$\lim_{s\to 0} \frac{\psi(1-s)}{\psi(s)} = \lim_{s\to 0} \frac{\frac{1}{2}s^{-\frac{1}{2}} - 1}{\frac{1}{2}(1-s)^{-\frac{1}{2}} - 1} = \frac{-1}{\frac{1}{2} - 1}$$

$= 2 < \infty$

(A3) Both $\psi(s)$ and $\psi(1\text{-}s)$ are regularly varying at zero with common index $\beta = 1 \in R$

$$\lim_{s\to 0} \frac{\psi(xs)}{\psi(s)} = \lim_{s\to 0} \frac{\frac{1}{2}(1-xs)^{-\frac{1}{2}} - 1}{\frac{1}{2}(1-s)^{-\frac{1}{2}} - 1} = \frac{-\frac{1}{2}}{-\frac{1}{2}} = 1$$

and

$$\lim_{s\to 0}\frac{\psi(1-xs)}{\psi(1-s)}=\lim_{s\to 0}\frac{\frac{1}{2}(xs)^{-\frac{1}{2}}-1}{\frac{1}{2}(s)^{-\frac{1}{2}}-1}=1$$

(A4) There exists a function $a(\cdot)$ not changing sign near zero such that

$$\lim_{s\to 0}\frac{\frac{\psi(xs)}{\psi(s)}-x^{\beta}}{a(s)}=x^{\beta}\frac{x^{\omega}-1}{x^{\omega}},\forall x>0$$

where $\omega \le 0$ is the second-order parameter. We will use this assumption to construct estimation for right-tailed deviation risk measure in next Section.

4. CONSTRUCTION OF ESTIMATION FOR WANG'S RIGHT-TAILED DEVIATION RISK MEASURES

4.1 Construction of a Risk Measure Estimator

We propose a risk measure by using the Wang's right-tailed deviation, as proposed in Section 3 and we define it as

$$\rho(X)=\int_0^1\left(\frac{1}{2\sqrt{1-s}}-1\right)F^{-1}(s)ds \tag{12}$$

where $F^{-1}(s) := \max\ \{x : F(x) \ge s\}$.

Generally, the distribution function F is unknown so $\rho(X)$ is also unknown. Suppose that we have $X_1, X_2, \ldots, X_n$ iid random variables with distribution function, F. We can estimate the risk measure by L-statistics as

$$\hat{\rho}(X)=\sum_{i=1}^{n}c_{i,n}X_{(i)} \tag{13}$$

where

$c_{i,n} := \int_{(i-1)/n}^{i/n}\psi(s)ds$, $i = 1, 2, \ldots, n$ and the $X_{(i)}$ are order statistics i.e., $X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$. Next, we recall the asymptotic normality properties of the estimator $\hat{\rho}(X)$.

Theorem 1

If ψ is a continuous function on (0,1) which satisfies the condition

$$|\psi(s)| \le cs^{\alpha-1}(1-s)^{\beta-1},\ 0 < s < 1 \tag{14}$$

for some constant $\alpha, \beta > 0$ and $c < \infty$, and if $E[|X|^{\gamma}] < \infty$ for some γ such that $\gamma < \frac{1}{\alpha}$ and $\gamma > \frac{1}{\beta}$, then

$$\hat{\rho}(X)\xrightarrow{\ p\ }\rho(X) \quad as\ n\to\infty \tag{15}$$

See Shorack and Wellner [26]

Theorem 2

If there exists $\gamma > 2$ such that $E[|X|^{\gamma}] < \infty$, then

$$\sqrt{n}(\hat{\rho}(X)-\rho(X))\xrightarrow{\ D\ }N(0,\sigma^2)\quad as\ n\to\infty \tag{16}$$

where D denotes convergence in distribution, and where the asymptotic variance σ^2 is given by

$$\sigma^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(F(x\wedge y)-F(x)F(y))\psi(F(x))\psi(F(y))dxdy \tag{17}$$

where $x \wedge y = \min(x,y)$

See Jones and Zitikis [1]

Otherwise, if the variance of distribution F is infinite then we cannot use the classical central limit theorem. It is of interest to construct a risk measure estimator and its confidence interval in case of the variance of the distribution function F is infinite. Here, we propose a new estimator based on the estimation of high quantiles.

We now define our estimator of our risk measure $\rho(X)$ by dividing $\rho(X)$ into three integrals as follows

$$\rho(X) = \int_0^{k/n} \psi(s)F^{-1}(s)ds + \int_{k/n}^{1-l/n} \psi(s)F^{-1}(s)ds$$

$$+ \int_{1-l/n}^{1} \psi(s)F^{-1}(s)ds \quad (18)$$

$$:= S_L + S_M + S_R$$

By substituting $\hat{F}_L^{-1}(s)$ and $\hat{F}_R^{-1}(s)$ for $F^{-1}(s)$ by $F^{-1}(1-s)$ in S_L and S_R respectively and using of assumption (A3), then

$$\int_0^{k/n} \psi(s)\hat{F}_L^{-1}(s)ds = \left(\frac{k}{n}\right)^{\hat{\gamma}_L} X_{(k)} \int_0^{k/n} s^{-\hat{\gamma}_L}\hat{F}_L^{-1}(s)ds$$

$$= (1+o(1))\frac{(k/n)\psi(k/n)}{2-\hat{\gamma}_L}X_{(k)},$$

and

$$\int_{1-l/n}^{1} \psi(1-s)\hat{F}_R^{-1}(1-s)ds = \left(\frac{l}{n}\right)^{\hat{\gamma}_R} X_{(n-l)} \int_{1-l/n}^{1} s^{-\hat{\gamma}_R}\hat{F}_R^{-1}(1-s)ds$$

$$= (1+o(1))\frac{(l/n)\psi(1-l/n)}{2-\hat{\gamma}_R}X_{(n-l)},$$

Thus, we can estimate S_L and S_R by

$$\hat{S}_L = \frac{(k/n)\psi(k/n)}{2-\hat{\gamma}_L}X_{(k)}, \quad (19)$$

and

$$\hat{S}_R = \frac{(l/n)\psi(1-l/n)}{2-\hat{\gamma}_R}X_{(n-l)}, \quad (20)$$

respectively. We take the sample one to estimate the S_M, that is

$$\hat{S}_M := \int_{k/n}^{1-l/n} \psi(s)\hat{F}_n^{-1}(s)ds = \sum_{i=k+1}^{n-l} c_{i,n}X_{(i)}, \quad (21)$$

with the same constant $c_{i,n}$ as appear in (13). Therefore, the our estimator is

$$\hat{\rho}(X) = \hat{S}_L + \hat{S}_M + \hat{S}_R$$

$$= \frac{(k/n)\psi(k/n)}{2-\hat{\gamma}_L}X_{(k)} + \sum_{i=k+1}^{n-l} c_{i,n}X_{(i)} +$$

$$\frac{(l/n)\psi(1-l/n)}{2-\hat{\gamma}_R}X_{(n-l)} \quad (22)$$

The asymptotic normality of our estimator is establish in the following theorem.

Theorem 3

Assume that $F\in D(\alpha)$, $0 < \alpha < 2$. for any measurable function ψ that satisfies assumption (A1-A4) with index $1 \in \mathbb{R}$ such that $\frac{3}{2} < \gamma < 2$, and for any sequences of integers k and l such that $1 < k < n, 1 < l < n$, $k \to \infty, l \to \infty, k/n \to 0, l/n \to 0, l/k \to \theta < \infty$ and $\sqrt{ka}\,(k/n)\, A(k/n)\, n \to 0$, as $n \to \infty$,

we have

$$\sqrt{n}\left(\hat{\rho}(X) - \rho(X)\right)/\sigma_n \xrightarrow{D} N\left(0, \sigma_0^2\right),$$

$as\ n \to \infty$

where

$$c_0^2 = \sigma_0^2(\alpha) := (\alpha+1)(\alpha+2)\left(\frac{2\alpha^2+(\alpha-1)^2+2\alpha(\alpha-1)}{2(2\alpha-1)^4} + \frac{1}{2\alpha-1}\right)+1$$

Corollary

Under the assumptions of theorem 3 we have

$$\frac{\sqrt{n}\left(\hat{\rho}(X) - \rho(X)\right)}{(l/n)^{1/2}\psi(1-l/n)X_{(n-l)}} \xrightarrow{D} N\left(0, V^2\right),$$

$as\ n \to \infty$

where

$$V^2 = V^2(\alpha,\theta,p) := \left(1+2^{-2}(q/p)^{-2/\alpha}\theta^{-3+2/\alpha}\right)$$

$$\times\left(\frac{2\alpha^2+(\alpha-1)^2+2\alpha(\alpha-1)}{2(2\alpha-1)^4} + \frac{1}{2\alpha-1}\right)+1,$$

with (p,q) as in statement (ii) of Section 2

4.2 Constructing Confidence Interval for Risk Measures

Suppose that, for n large enough, we have a realization $(x_1, x_2, \ldots, x_n)$ of a sample $(X_1, X_2, \ldots, X_n)$ from the random variable X with distribution function F that satisfies all assumptions of Theorem 3. The confidence interval for our risk measure will be computed as follows:

- Select the optimal integer m^* and of lower and upper order statistics that is in (8), (9), and (10).
- Find $X_{(k^*)}$, $X_{(n-l^*)}$, $\psi(k^* / n)$, $\psi(1 - l^* / n)$ and $\theta^* = l^* / k^*$.
- Calculate $\hat{\alpha}_L^* := \hat{\alpha}_L^*(k^*)$ and $\hat{\alpha}_R^* := \hat{\alpha}_R(l^*)$ by using (9). Then estimate $\hat{\rho}$
- Compute

$\hat{\alpha}^* := \hat{\alpha}(m^*)$ and $\hat{\rho}_n^* := \hat{\rho}_n(m^*)$ Then calculate the asymptotic standard deviation

$$V^* = \sqrt{V^2\left(\hat{\alpha}^*, \theta^*, \hat{p}_n^*\right)}$$

Therefore, the $(1-\eta)$% confidence interval of our risk measure will be estimated by

$$\hat{\rho}^* \pm z_{\eta/2} \frac{\sqrt{l^*} V^* X_{(n-l^*)} \psi(1-l^*)}{n}$$

where $z_{\eta/2}$ is the $(1-\eta / 2)$ quantile of the standard normal distribution $N(0,1)$ with $0 < \eta < 1$

5. CONCLUSIONS

In this paper, we investigated the estimation of a risk measure for heavy-tailed losses. The results are obtained for a specific risk measure, namely Wang's right-tailed deviation, but the techniques seem appropriate for other types of quantile-based risk measures. We intend to pursue this topic for general cases.

6. PROOFS

Proof of Theorem 3

Recall (18), (19), (20), (21), and (22) and write

$$\hat{\rho}(X) - \rho(X) = (\hat{S}_L - S_L) + (\hat{S}_M - S_M) + (\hat{S}_R - S_R) \quad (23)$$

We have

$$\hat{S}_L - S_L) = \frac{(k/n)\psi(k/n)X_{(k)}\hat{\alpha}_L}{2\hat{\alpha}_L - 1} - \int_0^{k/n} \psi(s)F^{1}(t)dt$$

$$= R_1^L + R_2^L + R_3^L \quad (24)$$

where

$$R_1^L := (k/n)\psi(k/n)X_{(k)}\left[\frac{\hat{\alpha}_L}{2\hat{\alpha}_L - 1} - \frac{\alpha}{2\alpha - 1}\right],$$

$$R_2^L := \frac{\alpha(k/n)F^{-1}(k/n)\psi(k/n)}{2\alpha - 1}\left[\frac{X_{(k)}}{F^{-1}(k/n)} - 1\right], \quad (25)$$

$$R_3^L := \frac{\alpha(k/n)F^{-1}(k/n)\psi(k/n)}{2\alpha - 1} - \int_0^{k/n} \psi(s)F^{-1}(t)dt$$

Likewise we have

$$\hat{S}_R - S_R = \frac{(l/n)\psi(1-l/n)X_{(n-l)}\hat{\alpha}_R}{2\hat{\alpha}_R - 1} - \int_0^{l/n} \psi(s)F^{-1}(1-t)dt$$

$$= R_1^R + R_2^R + R_3^R \quad (26)$$

where

$$R_1^R := (k/n)\psi(k/n)X_{(k)}\left[\frac{\hat{\alpha}_L}{2\hat{\alpha}_L - 1} - \frac{\alpha}{2\alpha - 1}\right]$$

$$R_2^R := \frac{\alpha(l/n)F^{-1}(1-l/n)\psi(1-l/n)}{2\alpha - 1}\left[\frac{X_{(n-l)}}{F^{-1}(1-l/n)} - 1\right] \quad (27)$$

$$R_3^R := \frac{\alpha(l/n)F^{-1}(1-l/n)\psi(1-l/n)}{2\alpha - 1} - \int_0^{l/n} \psi(s)F^{-1}(1-t)dt$$

R_1^L may be rewritten into

$$R_1^L = (1 + o_p(1))\frac{\alpha^2(k/n)\psi(k/n)X_{(k)}}{(2\alpha - 1)^2}\left[\frac{1}{\hat{\alpha}_L} - \frac{1}{\alpha}\right] \quad (28)$$

Since $\hat{\alpha}_L$ is a consistent estimator of α, then for all large n

$$R_1^L = (1 + o_p(1))\frac{\alpha^2(k/n)\psi(k/n)X_{(k)}}{(2\alpha - 1)^2}\left[\frac{1}{\hat{\alpha}_L} - \frac{1}{\alpha}\right] \quad (29)$$

In view of Theorem 2.3 and 2.4 of Csörgõ et al. [27], Peng [28] and Necir et al. [29] has been shown that under the second-order varying function and for all large n

$$\sqrt{k}\alpha\left[\frac{1}{\hat{\alpha}_L} - \frac{1}{\alpha}\right] = -\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right) + \sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds + o_p(1),$$

$$\sqrt{k}\left(\frac{X_{(k)}}{F^{-1}(k/n)} - 1\right) = -\alpha^{-1}\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right) + o_p(1),$$

$$\frac{X_{(k)}}{F^{-1}}\left(\frac{k}{n}\right) = 1 + o_p(1), \quad (30)$$

where $\{B_n(s), 0 \leq s \leq 1, n = 1,2,...\}$ is the sequence of Brownian bridge. This implies that for all large n

$$R_1^L = (1 + o_p(1))\frac{\alpha(\sqrt{k}/n)\psi(k/n)F^{-1}(k/n)}{(2\alpha - 1)^2}$$

$$\times\left[-\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right) + \sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds + o_p(1)\right],$$

$$R_2^L = \frac{\alpha(\sqrt{k}/n)\psi(k/n)F^{-1}(k/n)}{2\alpha - 1}\left[-\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right) + o_p(1)\right]$$

we have

$$\frac{\sqrt{n}\left(R_1^L+R_2^L\right)}{\sigma_n(\psi)}=\frac{\alpha\varpi}{(2\alpha-1)^2}\left[-\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+\sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds\right]$$

$$-\frac{\varpi}{2\alpha-1}\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+o_p(1) \qquad (31)$$

Likewise we have

$$\frac{\sqrt{n}\left(R_1^R+R_2^R\right)}{\sigma_n(\psi)}=\frac{\alpha\varpi_R}{(2\alpha-1)^2}\left[-\sqrt{\frac{n}{l}}B_n\left(1-\frac{l}{n}\right)+\sqrt{\frac{n}{l}}\int_{1-l/n}^{1}\frac{B_n(s)}{1-s}ds\right]$$

$$+\frac{\varpi_R}{2\alpha-1}\left[-\sqrt{\frac{n}{l}}B_n\left(1-\frac{l}{n}\right)\right]+o_p(1) \qquad (32)$$

where $\varpi_R := 2(q/p)^{1/\alpha}\theta^{3/2-1/\alpha}\varpi$. From the proof of Theorem 1 by Necir et al [29] yield that

$$\frac{\sqrt{n}R_3^R}{\sigma_n(\psi)}=\frac{\sqrt{n}R_3^R}{\sigma_n}(\psi)=o(1) \quad \text{as } n\to\infty \qquad (33)$$

Then, by (31), (32) and (33) we get

$$\frac{\sqrt{n}\left(\hat{\rho}(X)-\rho(X)\right)}{\sigma_n(\psi)}=\frac{\sqrt{n}\left(R_1^L+R_2^L\right)}{\sigma_n(\psi)}$$

$$=\frac{\alpha\varpi}{(2\alpha-1)^2}\left[-\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+\sqrt{\frac{n}{k}}\int_0^{k/n}\frac{B_n(s)}{s}ds\right]$$

$$-\frac{\varpi}{2\alpha-1}\sqrt{\frac{n}{k}}B_n\left(\frac{k}{n}\right)+o_p(1)-\frac{\int_{k/n}^{1-l/n}\psi(s)B_n(s)ds}{\sigma_n(\psi)}$$

$$+\frac{\varpi_R}{2\alpha-1}\left[-\sqrt{\frac{n}{l}}B_n\left(1-\frac{l}{n}\right)\right]+o_p(1)$$

$$+\frac{\alpha\varpi_R}{(2\alpha-1)^2}\left[-\sqrt{\frac{n}{l}}B_n\left(1-\frac{l}{n}\right)+\sqrt{\frac{n}{l}}\int_{1-l/n}^{1}\frac{B_n(s)}{1-s}ds\right] \qquad (34)$$

The asymptotic variance of $\dfrac{\sqrt{n}\left(\hat{\rho}(X)-\rho(X)\right)}{\sigma_n(\psi)}$ is computed by

$$\sigma_0^2=\lim_{n\to\infty}\Big[\varpi^2\frac{\alpha^2}{(2\alpha-1)^4}\frac{n}{k}\int_0^{k/n}ds\int_0^{k/n}\frac{s\wedge t-st}{st}dt$$

$$+\varpi^2(1-\alpha)^2\left(1-\frac{k}{n}\right)+\frac{\int_{k/n}^{1-l/n}dc(s)\int_{k/n}^{1-l/n}\frac{s\wedge t-st}{st}dc(s)}{\sigma_n(\psi)}$$

$$+\varpi_R^2(1-\alpha)^2\left(1-\frac{l}{n}\right)$$

$$+\varpi_R^2\frac{\alpha^2}{(2\alpha-1)^4}\frac{n}{l}\int_{1-l/n}^{1}ds\int_{1-l/n}^{1}\frac{s\wedge t-st}{st}dt$$

$$-2\varpi^2\alpha(1-\alpha)\frac{n}{k}\int_0^{k/n}\frac{t-(k/n)t}{t}dt$$

$$+2\varpi\frac{\alpha}{(2\alpha-1)^2}\sqrt{\frac{n}{k}}\int_0^{k/n}ds\int_{k/n}^{1-l/n}\frac{s-st}{s}dc(t)/\sigma_n(\psi)$$

$$-2\varpi\varpi_R\frac{\alpha(1-\alpha)}{(2\alpha-1)^4}\sqrt{\frac{n}{k}}\sqrt{\frac{n}{l}}\int_0^{k/n}\frac{s-(1-l/n)s}{s}ds$$

$$+2\varpi\varpi_R\frac{\alpha^2}{(2\alpha-1)^4}\sqrt{\frac{n}{k}}\sqrt{\frac{n}{l}}\int_0^{k/n}ds\int_{1-l/n}^{1}\frac{s\wedge t-st}{st}dt$$

$$-2\varpi\frac{1-\alpha}{(2\alpha-1)^2}\sqrt{\frac{n}{k}}\int_{k/n}^{1-l/n}\left(\frac{k}{n}-s\left(\frac{k}{n}\right)\right)dc(t)/\sigma_n(\psi)$$

$$+2\varpi\varpi_R\frac{(1-\alpha)^2}{(2\alpha-1)^4}\sqrt{\frac{n}{k}}\sqrt{\frac{n}{l}}\left(\frac{k}{n}-s\left(\frac{k}{n}\right)\right)$$

$$-2\varpi\varpi_R\frac{\alpha(1-\alpha)}{(2\alpha-1)^4}\int_{1-l/n}^{1}\left(\frac{k/n-(k/n)s}{1-s}\right)ds$$

$$-2\varpi\varpi_R\frac{1-\alpha}{(2\alpha-1)^2}\sqrt{\frac{n}{l}}\int_{k/n}^{1-l/n}\left(s-s\left(1-\frac{l}{n}\right)\right)dc(t)/\sigma_n(\psi)$$

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